Lecture 20: The connection between orthogonal polynomials and Riemann–Hilbert problems: proofs and outline of asymptotic calculation.

Lecture plan. We will carry out the proof that orthogonal polynomials can be characterized in terms of a Riemann–Hilbert problem, and then we will begin discussing the asymptotic analysis of Riemann–Hilbert problems.

Orthogonal Polynomials and Riemann–Hilbert problems

The following Riemann-Hilbert problem [1] is known to characterize the polynomials $p_j^{(N)}$ orthogonal with respect to $e^{-NV(x)}$.

Riemann-Hilbert Problem 1. *Find a* 2×2 *matrix* $\mathbf{A}(z) = \mathbf{A}(z; n, N)$ *with the properties:*

Analyticity. $\mathbf{A}(z)$ *is analytic for* $z \in \mathbb{C} \setminus \mathbb{R}$*, and takes continuous boundary values* $\mathbf{A}_+(x)$, $\mathbf{A}_-(x)$ *as* z *tends to* x *with* $x \in \mathbb{R}$ *and* $z \in \mathbb{C}_+$, $z \in \mathbb{C}_-$.

Jump Condition. *The boundary values are connected by the relation*

(1)
$$\mathbf{A}_+(x) = \mathbf{A}_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix} .$$

Normalization. *The matrix* $\mathbf{A}(z)$ *is normalized at* $z = \infty$ *as follows:*

(2)
$$\lim_{z \to \infty} \mathbf{A}(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = \mathbb{I} .$$

It was discovered in [1] that Riemann-Hilbert Problem 1 characterizes polynomials orthogonal with respect to $d\nu(x) := e^{-NV(x)}\, dx$. The connection between these orthogonal polynomials and the solution of Riemann-Hilbert Problem 1 is the following:

(3)
$$\mathbf{A}(z) = \begin{pmatrix} \dfrac{1}{\kappa_{n,n}^{(N)}} p_n(z) & \dfrac{1}{2\pi i \kappa_{n,n}^{(N)}} \displaystyle\int_{\mathbb{R}} \frac{p_n(s) e^{-NV(s)}}{s - z}\, ds \\ -2\pi i \kappa_{n-1,n-1}^{(N)} p_{n-1}(z) & -\kappa_{n-1,n-1}^{(N)} \displaystyle\int_{\mathbb{R}} \frac{p_{n-1}(s) e^{-NV(s)}}{s - z}\, ds \end{pmatrix} .$$

This relationship provides a useful avenue for asymptotic analysis of the orthogonal polynomials in the limit $n \to \infty$; it is sufficient to carry out a rigorous asymptotic analysis of Riemann-Hilbert Problem 1.

Uniqueness

The fact that there is at most one solution to Riemann–Hilbert problem 1 may be seen as follows. First you observe that if you have a solution, A, it is invertible. (You verify that $\det(A)(z)$ is entire and converges to 1 as $z \to \infty$, thus $\det(A)(z) \equiv 1$, by Liouville's theorem. Next you assume that there are two solutions, $A_1(z)$ and $A_2(z)$, and define

(4)
$$E(z) = A_1(z) A_2^{-1}(z).$$

You may easily verify that $E(z)$ is entire, and converges to I as $z \to \infty$, and then Liouville's theorem again takes over. This then implies that

(5)
$$A_1(z) = A_2(z)$$

and hence there is at most one solution to the Riemann–Hilbert problem.

Existence

The proof that the matrix (3) solves the Riemann–Hilbert problem 1 starts by observing that $A(z)$ defined in (3) automatically satisfies the analyticity condition, and the jump condition. To verify the jump condition, you need to know the following fact: if $F(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(s)}{s-z} ds$, and if $f(s)$ is a smooth and rapidly decaying function, then $F(z)$ is analytic for $z \in \mathbb{C}_+ \cup \mathbb{C}_-$, and possesses nice boundary values, $F_+(x)$ and $F_-(x)$ for $x \in \mathbb{R}$, which satisfy $F_+(x) - F_-(x) = f(x)$. (This can be extended in a variety of different directions - $\mathbb{R}$ can be replaced by a much more general contour, the function f need not be smooth, etc.)

So what remains to verify is the behavior for $z \to \infty$. The condition (2) implies that as $z \to \infty$,

$$A_{11}(z) = z^n + \text{lower order terms},$$
$$A_{21}(z) = cz^{n-1} + \text{lower order terms},$$

both being satisfied by (3). There are two other entries in the matrix A, and the condition (2) implies that as $z \to \infty$,

$$A_{12}(z) = cz^{-n-1} + \text{smaller terms},$$
$$A_{22}(z) = z^{-n} + \text{smaller terms}$$

Now at first glance these conditions are not satisfied by A_{12} and A_{22} as defined in (3). But upon closer inspection:

$$\begin{aligned} A_{12}(z) &= \frac{-1}{2\pi i \kappa_{n,n}^{(N)} z} \int_{\mathbb{R}} p_n(s) e^{-NV(s)} \left(\frac{1}{1 - s/z} \right) ds \\ &= \frac{-1}{2\pi i \kappa_{n,n}^{(N)} z} \int_{\mathbb{R}} p_n(s) e^{-NV(s)} \left(\sum_{j=0}^{n} \frac{s^j}{z^j} \right) ds + \frac{-1}{2\pi i \kappa_{n,n}^{(N)} z} \int_{\mathbb{R}} p_n(s) e^{-NV(s)} \left(\frac{s^{n+1}/z^{n+1}}{1 - s/z} \right) ds. \end{aligned}$$

Now by orthogonality, for $j = 0, 1, \ldots, n-1$, we know

$$\int_{\mathbb{R}} p_n(s) e^{-NV(s)} \left(\frac{s^j}{z^j} \right) ds = 0, \tag{6}$$

and also

$$\int_{\mathbb{R}} p_n(s) e^{-NV(s)} \left(s^n \right) ds = \int_{\mathbb{R}} p_n(s) e^{-NV(s)} \left(\frac{1}{\kappa_{n,n}^{(N)}} p_n(s) \right) ds = \frac{1}{\kappa_{n,n}^{(N)}}. \tag{7}$$

So, $A_{12}(z)$ defined in (3) satisfies

$$A_{12}(z) = -\frac{1}{2\pi i \left(\kappa_{n,n}^{(N)} \right)^2} z^{-n-1} + \mathcal{O}\left(z^{-n-2} \right). \tag{8}$$

We leave you all to check that $A_{22}(z)$ satisfies the remaining asymptotic condition:

$$A_{22}(z) = z^{-n} + \mathcal{O}\left(z^{-n-1} \right). \tag{9}$$

References

[1] A. Fokas, A. Its, and A. V. Kitaev, "Discrete Painlevé equations and their appearance in quantum gravity", *Commun. Math. Phys.*, **142**, 313–344, 1991.