

LECTURE 21: RIEMANN–HILBERT PROBLEMS AND SINGULAR INTEGRAL EQUATIONS, AND OUTLINE OF ASYMPTOTIC CALCULATION.

**Lecture plan.** For the discussion of the asymptotic analysis of Riemann–Hilbert problems, we will need to go through the connection to singular integral equations, and the theory of small norm Riemann–Hilbert problems.

ORTHOGONAL POLYNOMIALS AND RIEMANN–HILBERT PROBLEMS

Recall from the previous lecture that we have following Riemann-Hilbert problem which is known to characterize the polynomials  $p_j^{(N)}$  orthogonal with respect to  $e^{-NV(x)}$  [1].

**Riemann-Hilbert Problem 1.** Find a  $2 \times 2$  matrix  $\mathbf{A}(z) = \mathbf{A}(z; n, N)$  with the properties:

**Analyticity.**  $\mathbf{A}(z)$  is analytic for  $z \in \mathbb{C} \setminus \mathbb{R}$ , and takes continuous boundary values  $\mathbf{A}_+(x)$ ,  $\mathbf{A}_-(x)$  as  $z$  tends to  $x$  with  $x \in \mathbb{R}$  and  $z \in \mathbb{C}_+$ ,  $z \in \mathbb{C}_-$ .

**Jump Condition.** The boundary values are connected by the relation

$$(1) \quad \mathbf{A}_+(x) = \mathbf{A}_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix}.$$

**Normalization.** The matrix  $\mathbf{A}(z)$  is normalized at  $z = \infty$  as follows:

$$(2) \quad \lim_{z \rightarrow \infty} \mathbf{A}(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = \mathbb{I}.$$

The connection between these orthogonal polynomials and the solution of Riemann-Hilbert Problem 1 is the following:

$$(3) \quad \mathbf{A}(z) = \begin{pmatrix} \frac{1}{\kappa_{n,n}^{(N)}} p_n(z) & \frac{1}{2\pi i \kappa_{n,n}^{(N)}} \int_{\mathbb{R}} \frac{p_n(s) e^{-NV(s)}}{s-z} ds \\ -2\pi i \kappa_{n-1,n-1}^{(N)} p_{n-1}(z) & -\kappa_{n-1,n-1}^{(N)} \int_{\mathbb{R}} \frac{p_{n-1}(s) e^{-NV(s)}}{s-z} ds \end{pmatrix}.$$

This relationship provides a useful avenue for asymptotic analysis of the orthogonal polynomials in the limit  $n \rightarrow \infty$ ; it is sufficient to carry out a rigorous asymptotic analysis of Riemann-Hilbert Problem 1.

REPRESENTING THE KERNEL

The kernel  $K_N(x, y)$  can be represented directly in terms of the solution of the Riemann–Hilbert problem:

$$(4) \quad K_N(x, y) = -\frac{1}{2\pi i} e^{-\frac{N}{2}(V(x)+V(y))} \frac{Y_{11}(x)Y_{21}(y) - Y_{21}(x)Y_{11}(y)}{x-y} \\ = e^{-\frac{N}{2}(V(x)+V(y))} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

RIEMANN–HILBERT PROBLEMS, SINGULAR INTEGRAL EQUATIONS, AND SMALL NORM THEORY

The connection between Riemann–Hilbert problems and singular integral equations, as well as the related “small norm theory” is described in the lecture entitled “RHP Survey” available on the course website, after Lecture 21.

REFERENCES

[1] A. Fokas, A. Its, and A. V. Kitaev, “Discrete Painlevé equations and their appearance in quantum gravity”, *Commun. Math. Phys.*, **142**, 313–344, 1991.