

LECTURE 23: RIEMANN–HILBERT PROBLEMS AND SINGULAR INTEGRAL EQUATIONS, AND OUTLINE OF ASYMPTOTIC CALCULATION.

Lecture plan. The goal of this lecture is to explain the steps involved with computing the asymptotic analysis of the Riemann–Hilbert problem associated to orthogonal polynomials. Having defined the equilibrium measure, we will introduce the first transformation of the Riemann–Hilbert problem, and then proceed with an outline of the asymptotic analysis.

USING EQUILIBRIUM MEASURES

Recall that the equilibrium measure is defined as the unique minimizer in

$$(1) \quad \mathcal{M}_1(\mathbb{R} = \{\text{Probability measures on } \mathbb{R}\})$$

of the functional

$$(2) \quad I_\beta : \mathcal{M}_1 \rightarrow (-\infty, \infty], \mu \mapsto \int_{\mathbb{R}^2} \log|x-y|^{-1} d\mu(x)d\mu(y) + \int_{\mathbb{R}} \kappa_\beta |x|^\beta d\mu(x).$$

In the previous lecture we discussed the various origins of this variational problem, and how it relates to orthogonal polynomials and random matrix theory. In this lecture we will require (later) the following properties of the equilibrium measure

- The equilibrium measure μ^* is a.c. w.r.t. Lebesgue measure, $\int f(x)d\mu^*(x) = \int_{-1}^1 f(x)\psi_\beta(x)dx$, and

$$(3) \quad \psi_\beta(x) = \frac{\beta}{\pi} \int_{|x|}^1 \frac{u^{\beta-1}}{\sqrt{u^2-x^2}} du, \quad x \in (-1, 1).$$

- There is a constant ℓ so that for all $x \in (-1, 1)$, the equilibrium measure satisfies

$$(4) \quad 2 \int_{-1}^1 \log|x-y|d\mu^*(y) - V(x) = \ell$$

- For $x \in \mathbb{R} \setminus [-1, 1]$, the following holds:

$$(5) \quad 2 \int_{-1}^1 \log|x-y|d\mu^*(y) - V(x) < \ell$$

FIRST TRANSFORMATION OF THE RIEMANN–HILBERT PROBLEM

Recall from the previous lecture that we have following Riemann-Hilbert problem which is known to characterize the polynomials $p_j^{(N)}$ orthogonal with respect to $e^{-NV(x)}$ [1].

Riemann-Hilbert Problem 1. Find a 2×2 matrix $\mathbf{A}(z) = \mathbf{A}(z; n, N)$ with the properties:

Analyticity. $\mathbf{A}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$, and takes continuous boundary values $\mathbf{A}_+(x)$, $\mathbf{A}_-(x)$ as z tends to x with $x \in \mathbb{R}$ and $z \in \mathbb{C}_+$, $z \in \mathbb{C}_-$.

Jump Condition. The boundary values are connected by the relation

$$(6) \quad \mathbf{A}_+(x) = \mathbf{A}_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix}.$$

Normalization. The matrix $\mathbf{A}(z)$ is normalized at $z = \infty$ as follows:

$$(7) \quad \lim_{z \rightarrow \infty} \mathbf{A}(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = \mathbb{I}.$$

The connection between these orthogonal polynomials and the solution of Riemann-Hilbert Problem 1 is the following:

$$(8) \quad \mathbf{A}(z) = \begin{pmatrix} \frac{1}{\kappa_{n,n}^{(N)}} p_n(z) & \frac{1}{2\pi i \kappa_{n,n}^{(N)}} \int_{\mathbb{R}} \frac{p_n(s) e^{-NV(s)}}{s-z} ds \\ -2\pi i \kappa_{n-1,n-1}^{(N)} p_{n-1}(z) & -\kappa_{n-1,n-1}^{(N)} \int_{\mathbb{R}} \frac{p_{n-1}(s) e^{-NV(s)}}{s-z} ds \end{pmatrix}.$$

This relationship provides a useful avenue for asymptotic analysis of the orthogonal polynomials in the limit $n \rightarrow \infty$; it is sufficient to carry out a rigorous asymptotic analysis of Riemann-Hilbert Problem 1.

The first transformation is as follows. Define

$$(9) \quad g(z) = \int_{-1}^1 \log(z-x) d\mu^*(x) = \int_{-1}^1 \log(z-x) \psi_\beta(x) dx$$

which is taken to be analytic in $\mathbb{C} \setminus (-\infty, 1]$. Using $g(z)$, we define a new matrix valued function (the new unknown) $B(z)$, as follows:

$$(10) \quad \mathbf{B}(z) := e^{-\frac{N\ell}{2}\sigma_3} \mathbf{A}(z) e^{-N(g(z) - \frac{\ell}{2})\sigma_3}$$

We will verify that \mathbf{B} satisfies a new Riemann-Hilbert problem:

Riemann-Hilbert Problem 2. Find a 2×2 matrix $\mathbf{B}(z) = \mathbf{B}(z; n, N)$ with the properties:

Analyticity. $\mathbf{B}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$, and takes continuous boundary values $\mathbf{B}_+(x)$, $\mathbf{B}_-(x)$ as z tends to x with $x \in \mathbb{R}$ and $z \in \mathbb{C}_+$, $z \in \mathbb{C}_-$.

Jump Condition. The boundary values are connected by the relation

$$(11) \quad \mathbf{B}_+(x) = \mathbf{B}_-(x) \begin{pmatrix} e^{-N(g_+(x) - g_-(x))} & e^{N(g_+(x) + g_-(x) - V(x) - \ell)} \\ 0 & e^{N(g_+(x) - g_-(x))} \end{pmatrix}.$$

Normalization. The matrix $\mathbf{B}(z)$ is normalized at $z = \infty$ as follows:

$$(12) \quad \lim_{z \rightarrow \infty} \mathbf{B}(z) = \mathbb{I}.$$

Here is a very useful result concerning the function g defined, as used, above:

Proposition 3.6. There exists a $\delta_1 > 0$ such that for all $n \in \mathbb{N}$, the following holds.

(a) g is analytic and $g|_{\mathbb{C}_\pm}$ have continuous extensions to $\overline{\mathbb{C}_\pm}$.

(b) The map $z \mapsto e^{ng(z)}$ possesses an analytic continuation to $\mathbb{C} \setminus [-1, 1]$ and

$$e^{ng(z)} z^{-n} = 1 + \mathcal{O}\left(\frac{1}{|z|}\right) \quad \text{as } z \rightarrow \infty. \quad (3.7)$$

(c)

$$g_+(x) - g_-(x) = \begin{cases} 2\pi i & \text{for } x \leq -1, \\ 2\pi i \int_x^1 \psi_\beta(s) ds & \text{for } |x| < 1, \\ 0 & \text{for } x \geq 1. \end{cases} \quad (3.8)$$

The function $g_+ - g_-$ possesses an analytic continuation G to the strips $\mathcal{S}_\pm \equiv \{z \in \mathbb{C}: 0 < \operatorname{Re}(\pm z) < 1, |\operatorname{Im}z| < \delta_1\}$ such that

$$\operatorname{Re} G(z) > 0 \quad \text{for } z \in \mathbb{C}_+ \cap \mathcal{S}_\pm, \quad (3.9)$$

$$\operatorname{Re} G(z) < 0 \quad \text{for } z \in \mathbb{C}_- \cap \mathcal{S}_\pm. \quad (3.10)$$

(d)

$$g_+(x) + g_-(x) - \kappa_\beta |x|^\beta - \ell = \begin{cases} -2\beta \int_1^{|x|} \left(\int_1^s \frac{u^{\beta-1}}{(s^2-u^2)^{1/2}} du \right) ds & \text{for } x \leq -1, \\ 0 & \text{for } |x| < 1, \\ -2\beta \int_1^x \left(\int_1^s \frac{u^{\beta-1}}{(s^2-u^2)^{1/2}} du \right) ds & \text{for } x \geq 1. \end{cases} \quad (3.11)$$

□

REFERENCES

- [1] A. Fokas, A. Its, and A. V. Kitaev, “Discrete Painlevé equations and their appearance in quantum gravity”, *Commun. Math. Phys.*, **142**, 313–344, 1991.