

LECTURE 24: CONTINUING THE ASYMPTOTIC CALCULATION.

Lecture plan. Starting again with the first transformation of the RHP, we will proceed with an outline of the asymptotic analysis.

USING EQUILIBRIUM MEASURES

Recall that the equilibrium measure is defined as the unique minimizer in

$$(1) \quad \mathcal{M}_1(\mathbb{R} = \{\text{Probability measures on } \mathbb{R}\})$$

of the functional

$$(2) \quad I_\beta : \mathcal{M}_1 \rightarrow (-\infty, \infty], \mu \mapsto \int_{\mathbb{R}^2} \log|x-y|^{-1} d\mu(x)d\mu(y) + \int_{\mathbb{R}} \kappa_\beta|x|^\beta d\mu(x).$$

In the previous lecture we discussed the various origins of this variational problem, and how it relates to orthogonal polynomials and random matrix theory. In this lecture we will require (later) the following properties of the equilibrium measure

- The equilibrium measure μ^* is a.c. w.r.t. Lebesgue measure, $\int f(x)d\mu^*(x) = \int_{-1}^1 f(x)\psi_\beta(x)dx$, and

$$(3) \quad \psi_\beta(x) = \frac{\beta}{\pi} \int_{|x|}^1 \frac{u^{\beta-1}}{\sqrt{u^2-x^2}} du, \quad x \in (-1, 1).$$

- There is a constant ℓ so that for all $x \in (-1, 1)$, the equilibrium measure satisfies

$$(4) \quad 2 \int_{-1}^1 \log|x-y|d\mu^*(y) - V(x) = \ell$$

- For $x \in \mathbb{R} \setminus [-1, 1]$, the following holds:

$$(5) \quad 2 \int_{-1}^1 \log|x-y|d\mu^*(y) - V(x) < \ell$$

FIRST TRANSFORMATION OF THE RIEMANN–HILBERT PROBLEM

Recall from the previous lecture that we have following Riemann-Hilbert problem which is known to characterize the polynomials $p_j^{(N)}$ orthogonal with respect to $e^{-NV(x)}$ [3]. The calculations of the present lecture were first carried out in [1] and [2], but for the particular case of orthogonality with respect to $e^{-N\kappa_\beta|x|^\beta} dx$, the work was presented in [4]

Riemann-Hilbert Problem 1. Find a 2×2 matrix $\mathbf{A}(z) = \mathbf{A}(z; n, N)$ with the properties:

Analyticity. $\mathbf{A}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$, and takes continuous boundary values $\mathbf{A}_+(x)$, $\mathbf{A}_-(x)$ as z tends to x with $x \in \mathbb{R}$ and $z \in \mathbb{C}_+$, $z \in \mathbb{C}_-$.

Jump Condition. The boundary values are connected by the relation

$$(6) \quad \mathbf{A}_+(x) = \mathbf{A}_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix}.$$

Normalization. The matrix $\mathbf{A}(z)$ is normalized at $z = \infty$ as follows:

$$(7) \quad \lim_{z \rightarrow \infty} \mathbf{A}(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = \mathbb{I}.$$

The connection between these orthogonal polynomials and the solution of Riemann-Hilbert Problem 1 is the following:

$$(8) \quad \mathbf{A}(z) = \begin{pmatrix} \frac{1}{\kappa_{n,n}^{(N)}} p_n(z) & \frac{1}{2\pi i \kappa_{n,n}^{(N)}} \int_{\mathbb{R}} \frac{p_n(s) e^{-NV(s)}}{s-z} ds \\ -2\pi i \kappa_{n-1,n-1}^{(N)} p_{n-1}(z) & -\kappa_{n-1,n-1}^{(N)} \int_{\mathbb{R}} \frac{p_{n-1}(s) e^{-NV(s)}}{s-z} ds \end{pmatrix}.$$

This relationship provides a useful avenue for asymptotic analysis of the orthogonal polynomials in the limit $n \rightarrow \infty$; it is sufficient to carry out a rigorous asymptotic analysis of Riemann-Hilbert Problem 1.

The first transformation is as follows. Define

$$(9) \quad g(z) = \int_{-1}^1 \log(z-x) d\mu^*(x) = \int_{-1}^1 \log(z-x) \psi_\beta(x) dx$$

which is taken to be analytic in $\mathbb{C} \setminus (-\infty, 1]$. Using $g(z)$, we define a new matrix valued function (the new unknown) $B(z)$, as follows:

$$(10) \quad \mathbf{B}(z) := e^{-\frac{N\ell}{2}\sigma_3} \mathbf{A}(z) e^{-N(g(z) - \frac{\ell}{2})\sigma_3}$$

We will verify that \mathbf{B} satisfies a new Riemann-Hilbert problem:

Riemann-Hilbert Problem 2. Find a 2×2 matrix $\mathbf{B}(z) = \mathbf{B}(z; n, N)$ with the properties:

Analyticity. $\mathbf{B}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$, and takes continuous boundary values $\mathbf{B}_+(x)$, $\mathbf{B}_-(x)$ as z tends to x with $x \in \mathbb{R}$ and $z \in \mathbb{C}_+$, $z \in \mathbb{C}_-$.

Jump Condition. The boundary values are connected by the relation

$$(11) \quad \mathbf{B}_+(x) = \mathbf{B}_-(x) \begin{pmatrix} e^{-N(g_+(x) - g_-(x))} & e^{N(g_+(x) + g_-(x) - V(x) - \ell)} \\ 0 & e^{N(g_+(x) - g_-(x))} \end{pmatrix}.$$

Normalization. The matrix $\mathbf{B}(z)$ is normalized at $z = \infty$ as follows:

$$(12) \quad \lim_{z \rightarrow \infty} \mathbf{B}(z) = \mathbb{I}.$$

Here is a very useful result concerning the function g defined, as used, above:

Proposition 3.6. There exists a $\delta_1 > 0$ such that for all $n \in \mathbb{N}$, the following holds.

- (a) g is analytic and $g|_{\mathbb{C}_\pm}$ have continuous extensions to $\overline{\mathbb{C}_\pm}$.
- (b) The map $z \mapsto e^{ng(z)}$ possesses an analytic continuation to $\mathbb{C} \setminus [-1, 1]$ and

$$e^{ng(z)} z^{-n} = 1 + \mathcal{O}\left(\frac{1}{|z|}\right) \quad \text{as } z \rightarrow \infty. \quad (3.7)$$

(c)

$$g_+(x) - g_-(x) = \begin{cases} 2\pi i & \text{for } x \leq -1, \\ 2\pi i \int_x^1 \psi_\beta(s) ds & \text{for } |x| < 1, \\ 0 & \text{for } x \geq 1. \end{cases} \quad (3.8)$$

The function $g_+ - g_-$ possesses an analytic continuation G to the strips $\mathcal{S}_\pm \equiv \{z \in \mathbb{C} : 0 < \operatorname{Re}(\pm z) < 1, |\operatorname{Im}z| < \delta_1\}$ such that

$$\operatorname{Re} G(z) > 0 \quad \text{for } z \in \mathbb{C}_+ \cap \mathcal{S}_\pm, \quad (3.9)$$

$$\operatorname{Re} G(z) < 0 \quad \text{for } z \in \mathbb{C}_- \cap \mathcal{S}_\pm. \quad (3.10)$$

(d)

$$g_+(x) + g_-(x) - \kappa_\beta |x|^\beta - \ell = \begin{cases} -2\beta \int_1^{|x|} \left(\int_1^s \frac{u^{\beta-1}}{(s^2-u^2)^{1/2}} du \right) ds & \text{for } x \leq -1, \\ 0 & \text{for } |x| < 1, \\ -2\beta \int_1^x \left(\int_1^s \frac{u^{\beta-1}}{(s^2-u^2)^{1/2}} du \right) ds & \text{for } x \geq 1. \end{cases} \quad (3.11)$$

□

Using the properties of the function g outlined above, it is straightforward to verify that the jump matrix for B takes one of the following forms:

$$\left\{ \begin{array}{l} \left(\begin{array}{cc} e^{-nG(s)} & 1 \\ 0 & e^{nG(s)} \end{array} \right) \quad \text{for } -1 < s < 1, \\ \left(\begin{array}{cc} 1 & e^{n(g_+(s)+g_-(s)-\kappa_\beta |s|^\beta - \ell)} \\ 0 & 1 \end{array} \right) \quad \text{for } |s| \geq 1. \end{array} \right.$$

Where the function G is defined above. It is purely imaginary on the interval $(-1, 1)$. Check out the factorization:

$$\begin{pmatrix} e^{-nG(s)} & 1 \\ 0 & e^{nG(s)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{nG(s)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-nG(s)} & 1 \end{pmatrix} \equiv \mathbf{v}_- \mathbf{v}_0 \mathbf{v}_+$$

Now the amazing thing is that these matrices can be extended off the real axis, where the oscillations turn into *decay*. The two relevant statements are these:

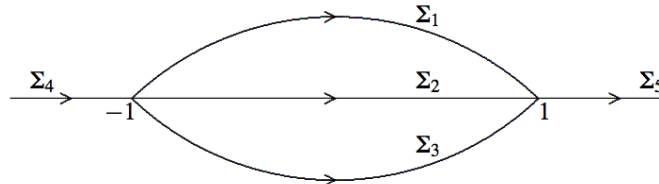
- $e^{nG(z)}$ can be extended analytically *below* $(-1, 1)$, where it is exponentially small as $n \rightarrow \infty$.
- $e^{nG(z)}$ can be extended analytically *above* $(-1, 1)$, where it is exponentially small as $n \rightarrow \infty$.

These two facts may be verified as follows: first, you check that $\text{Re}(G(z)) = 0$ for $z \in (-1, 1)$, and then by the Cauchy-Riemann equations,

$$(13) \quad \left. \frac{\partial}{\partial y} \text{Re}(G(x+iy)) \right|_{y=0} = - \left. \frac{\partial}{\partial x} \text{Im}(G(x+iy)) \right|_{y=0} = 2\pi\psi(x) > 0.$$

So we then define $D(z)$ as follows:

- For z outside the “lens shaped region” surrounding the interval $(-1, 1)$, $D(z) = B(z)$.
- For z within the “upper lens shaped region”, we set $D(z) = B(z)v_+(z)^{-1}$.
- For z within the “lower lens shaped region”, we set $D(z) = B(z)v_-(z)$.



And it is clear that the new unknown, D , is analytic of the more complicated union of contours shown above. Moreover, given the above considerations, the jump matrices satisfy the following important property:

For any $\delta > 0$, the jump matrix $V_D(z)$ is exponentially close to \mathbb{I} for all values of z whose distance from $[-1, 1]$ is greater than δ .

Homework Problem: Ignoring all contours except the interval $[-1, 1]$, solve the following Riemann–Hilbert problem.

Riemann-Hilbert Problem 3. Find $\dot{D}(z)$ satisfying the following three conditions.

- (1) (analyticity) The matrix \dot{D} is analytic in $\mathbb{C} \setminus [-1, 1]$.
- (2) (Normalization) $\dot{D}(z) = \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \rightarrow \infty$.
- (3) (Boundary values and jump relation)

$$(14) \quad \dot{D}_+(x) = \dot{D}_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

REFERENCES

- [1] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, “Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory”, *Comm. Pure Appl. Math.*, **52**, 1335–1425, 1999.
- [2] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, “Strong asymptotics of orthogonal polynomials with respect to exponential weights”, *Comm. Pure Appl. Math.*, **52**, 1491–1552, 1999.
- [3] A. Fokas, A. Its, and A. V. Kitaev, “Discrete Painlevé equations and their appearance in quantum gravity”, *Commun. Math. Phys.*, **142**, 313–344, 1991.
- [4] T. Kriecherbauer and K. T.-R. McLaughlin, Strong Asymptotics of Polynomials Orthogonal with Respect to Freud Weights, *Int. Math. Res. Not.*, No. 6, pp. 299333, 1999.