Lecture 26: The Airy functions and their connection to Riemann–Hilbert problems

Lecture plan. We will summarize how we have completed the asymptotic calculation, followed by an explanation of how we can build the local parametrix out of Airy functions.

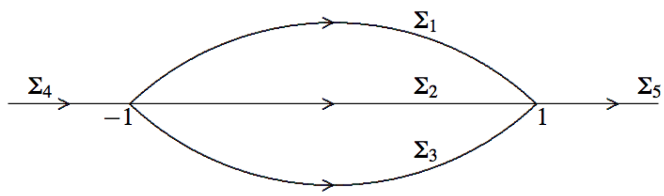
summary of the asymptotic calculation

We began with $\mathbf{A}(z)$ solving the original Riemann–Hilbert problem for orthogonal polynomials. The first transformation was

(1) $$\mathbf{B}(z) := e^{-\frac{N\ell}{2}\sigma_3}\mathbf{A}(z)e^{-N\left(g(z)-\frac{\ell}{2}\right)\sigma_3}.$$

The second transformation was as follows:
So we then defined $\mathbf{D}(z)$ as follows:

- For z outside the "lens shaped region" surrounding the interval $(-1,1)$, $\mathbf{D}(z) = B(z)$.
- For z within the "upper lens shaped region", we set $\mathbf{D}(z) = B(z)v_+(z)^{-1}$.
- For z within the "lower lens shaped region", we set $\mathbf{D}(z) = B(z)v_-(z)$.



The matrix D now solves a new Riemann–Hilbert problem.

Riemann-Hilbert Problem 1. *Find $D(z)$, satisfying the following three conditions.*

Analyticity. *$\mathbf{D}(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$, and takes continuous boundary values $\mathbf{D}_+(z)$, $\mathbf{D}_-(z)$ with $x \in \Sigma$.*

Jump Condition. *The boundary values are connected by the relation*

(2) $$\mathbf{D}_+(z) = \mathbf{D}_-(z)V_D(z) \quad , z \in \Sigma$$

Normalization. *The matrix $\mathbf{D}(z)$ is normalized at $z = \infty$ as follows:*

(3) $$\lim_{z\to\infty} \mathbf{D}(z) = \mathbb{I}\,.$$

The jump matrix V_D was defined as follows:

- *For $z \in \Sigma_4 \cup \Sigma_5$, we have $V_D(z) = V_B(z)$.*
- *For $z \in \Sigma_2$, we have*

(4) $$V_D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- *For $z \in \Sigma_1$, we have $V_D(z) = v_+(z)$.*
- *For $z \in \Sigma_3$, we have $V_D(z) = v_-(z)$.*

And it is clear that the new unknown, D, is analytic of the more complicated union of contours shown above. Moreover, given the above considerations, the jump matrices satisfy the following important property:

For any $\delta > 0$, the jump matrix $V_D(z)$ is exponentially close to $\mathbb{I}$ for all values of z whose distance from $[-1,1]$ is greater than δ.

global approximation to D. We then began building a global approximation to $\mathbf{D}$, by asking if we could find $\dot{D}$ solving the following *reduced* Riemann–Hilbert problem:

Riemann-Hilbert Problem 2. *Find $\dot{D}(z)$ satisfying the following three conditions.*

(1) *(analyticity) The matrix $\dot{D}$ is analytic in $\mathbb{C} \setminus [-1,1]$.*
(2) *(Normalization) $\dot{D}(z) = \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.*
(3) *(Boundary values and jump relation)*

$$\dot{D}_+(x) = \dot{D}_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{5}$$

The function $\dot{D}$ is explicitly known:

$$\dot{D}(z) = \mathbb{F} \begin{pmatrix} a(z) & 0 \\ 0 & \frac{1}{a(z)} \end{pmatrix} \mathbb{F}^{-1} \tag{6}$$

where $\mathbb{F}$ is the following explicit matrix:

$$\mathbb{F} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} . \tag{7}$$

The intuition which we have developed indicates that

$$\mathbf{E}(z) := \mathbf{D}(z) \ \left(\dot{D}(z)\right)^{-1} \tag{8}$$

should be a new unknown which has *no jump across* $(-1,1)$, and has jumps that are exponentially near to $\mathbb{I}$ for z in the contour Σ but bounded away from ± 1, and so maybe this quantity satisfies the *guiding principle* we have spoken about in Lecture 21 (and see the discussion in the "RHPSurvey" lecture posted on the website).

But we are not quite home yet: the jump matrices are not uniformly near to $\mathbb{I}$. In Lecture 25 we began discussing the local behavior of the jump matrices in a vicinity of $z = 1$. The end result was that under the transformation

$$\frac{4}{3}\zeta(z)^{3/2} = 2\pi i N \int_1^x \tilde{\psi}_\beta(s) ds \tag{9}$$

We (more or less) verified that $\zeta(z)$ is analytic for z in a neighborhood of $z = 1$, and maps a disc of fixed size centered at $z = 1$ to a neighborhood of $\zeta = 0$. The neighborhood in the ζ-plane is very large (roughly $\mathcal{O}(N^{2/3})$).

The point to this transformation is that in the new ζ-plane, the jump matrices $V_D(z(\zeta))$ take on the following form:

$$\hat{v}^{(1)}(\zeta) = \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix}, \quad \arg\zeta = 0,$$

$$\hat{v}^{(1)}(\zeta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \arg\zeta = \pi,$$

$$\hat{v}^{(1)}(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, \quad \arg\zeta = \frac{2\pi}{3},$$

$$\hat{v}^{(1)}(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, \quad \arg\zeta = \frac{4\pi}{3}.$$

Now the idea here is as follows: *can we construct a matrix valued function which has exactly these jumps in the ζ plane? If so, can we port it back over to the z plane, and use it as a "parametrix" in a neighborhood of $z = 1$?* Let's call this function $P_1(\zeta)$.

Similar calculations may be carried out in a vicinity of $z = -1$, and for those calculations, we will let the analogous function be referred to as $P_{-1}(\zeta)$.

So now we have built three separate matrix valued functions which together may be used as a global approximation to $\mathbf{D}(z)$. Here is the definition:

- For all $z \in \mathbb{C}$ but *outside* discs of radius $\delta > 0$ centered at $z = \pm 1$, we define

$$\mathbf{D_a}(z) = \dot{D}(z). \tag{10}$$

- For $|z-1| < \delta$, we define

$$\mathbf{D_a}(z) = \mathcal{A}_1(z) P_1(\zeta(z)). \tag{11}$$

- For $|z+1| < \delta$, we define

$$\mathbf{D_a}(z) = \mathcal{A}_{-1}(z) P_{-1}(\zeta(z)). \tag{12}$$

NOTE: The matrices $\mathcal{A}_1$ and $\mathcal{A}_{-1}$ are matrices which are still yet to be determined. They will be *analytic* in a vicinity of 1 and -1, respectively.

How good is the global approximation

As we have seen in previous lectures, the only way to assess the approximation $\mathbf{D_a}$ is to consider the ratio

$$\mathbf{E}(z) := \mathbf{D}(z) \cdot \mathbf{D_a}(z)^{-1}, \tag{13}$$

which solves yet another Riemann–Hilbert problem.

Riemann-Hilbert Problem 3. *Find $\mathbf{E}(z)$, satisfying the following three conditions.*

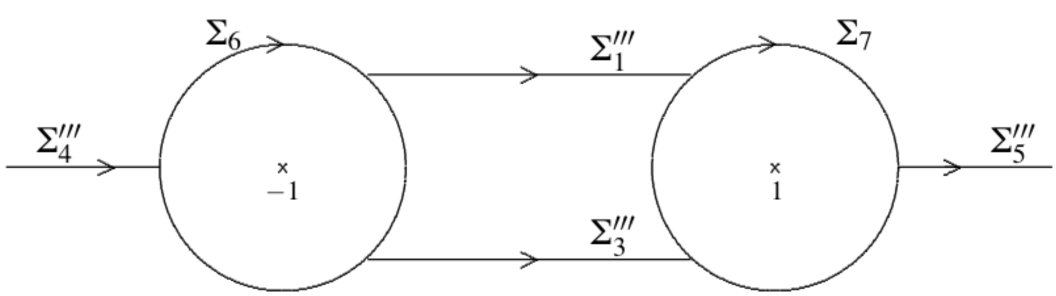
Analyticity. *$\mathbf{E}(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma_E$, and takes continuous boundary values $\mathbf{E}_+(z)$, $\mathbf{E}_-(z)$ with $x \in \Sigma$.*

Jump Condition. *The boundary values are connected by the relation*

$$\mathbf{E}_+(z) = \mathbf{E}_-(z) V_E(z) \quad , z \in \Sigma_E \tag{14}$$

Normalization. *The matrix $\mathbf{E}(z)$ is normalized at $z = \infty$ as follows:*

$$\lim_{z \to \infty} \mathbf{E}(z) = \mathbb{I}\,. \tag{15}$$



The jump matrix V_E is defined as follows:

- *For $z \in \Sigma_4''' \cup \Sigma_5'''$, we have $V_E(z) = V_B(z) = \mathbb{I} + \mathcal{O}\left(\left|e^{-2\pi i N \int_1^z \tilde{\psi}_\beta(s)ds}\right|\right)$.*
- *For $z \in \Sigma_1'''$, we have $V_E(z) = v_+(z) = \mathbb{I} + \mathcal{O}\left(e^{-cN}\right)$.*
- *For $z \in \Sigma_3'''$, we have $V_E(z) = v_-(z) = \mathbb{I} + \mathcal{O}\left(e^{-cN}\right)$.*
- *For $z \in \Sigma_7$, we have $V_E(z) = \mathcal{A}_1(z) P_1(\zeta(z)) \dot{D}(z)^{-1}$.*
- *For $z \in \Sigma_6$, we have $V_E(z) = \mathcal{A}_{-1}(z) P_{-1}(\zeta(z)) \dot{D}(z)^{-1}$.*

The claim, left to the lecture today, is that we can build the local "parametrices" P_1 and P_{-1}, along with the analytic pre-factors $\mathcal{A}_1$ and $\mathcal{A}_{-1}$, so that the jumps on the contours Σ_6 and Σ_7 are, finally, $\mathbb{I} + \mathcal{O}\left(N^{-1}\right)$.

So we consider a full-fledged Riemann–Hilbert problem for $P_1(\zeta)$, in the ζ-plane.

Riemann-Hilbert Problem 4. *Find $P_1(\zeta)$, satisfying the following three conditions.*

Analyticity. *$P_1(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus \Sigma_{P_1}$, and takes continuous boundary values $(P_1)_+(\zeta)$, $(P_1)_-(\zeta)$ with $x \in \Sigma$.*

Jump Condition. *The boundary values are connected by the relation*

(16)
$$(P_1)_+(\zeta) = (P_1)_-(\zeta)\hat{v}^{(1)}(\zeta) \ , \zeta \in \Sigma_{P_1}$$

Normalization. *The matrix $P_1(\zeta)$ is normalized at $\zeta = \infty$ as follows:*

$$P(\zeta) = \zeta^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3}\left(I + O(\zeta^{-3/2})\right), \quad \zeta \to \infty,$$

The jump matrix $\hat{v}^{(1)}(\zeta)$ is defined as follows:

$$\hat{v}^{(1)}(\zeta) = \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix}, \quad \arg\zeta = 0,$$

$$\hat{v}^{(1)}(\zeta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \arg\zeta = \pi,$$

$$\hat{v}^{(1)}(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, \quad \arg\zeta = \frac{2\pi}{3},$$

$$\hat{v}^{(1)}(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, \quad \arg\zeta = \frac{4\pi}{3}.$$

We first define the matrix-valued function $\Psi(\zeta)$ as follows:

(17)
$$\mathbf{\Psi}(\zeta) = \begin{pmatrix} Ai(\zeta) & Ai(e^{-2i\pi/3}\zeta) \\ Ai'(\zeta) & e^{-2i\pi/3}Ai'(e^{-2i\pi/3}\zeta) \end{pmatrix}, \quad \text{for } 0 < \arg(\zeta) < \pi,$$

(18)
$$\mathbf{\Psi}(\zeta) = \begin{pmatrix} Ai(\zeta) & -e^{-2i\pi/3}Ai(e^{2i\pi/3}\zeta) \\ Ai'(\zeta) & -Ai'(e^{2i\pi/3}\zeta) \end{pmatrix}, \quad \text{for } -\pi < \arg(\zeta) < 0,$$

where $\omega = e^{2i\pi/3}$. Using $\mathbf{\Psi}$, we now define P_1 as follows:

(19)
$$P_1(\zeta) = \sqrt{2\pi}e^{\frac{-i\pi}{12}} \ \mathbf{\Psi}(\zeta)e^{\left(\frac{2}{3}\zeta^{3/2} - \frac{i\pi}{6}\right)\sigma_3} \ \text{ for } |\arg(\zeta)| < \frac{2\pi}{3},$$

(20)
$$P_1(\zeta) = \sqrt{2\pi}e^{\frac{-i\pi}{12}}\mathbf{\Psi}(\zeta)e^{\left(\frac{2}{3}\zeta^{3/2} - \frac{i\pi}{6}\right)\sigma_3} \begin{pmatrix} 1 & 0 \\ -e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix} \ \text{ for } \frac{2\pi}{3} < \arg(\zeta) < \pi,$$

(21)
$$P_1(\zeta) = \sqrt{2\pi}e^{\frac{-i\pi}{12}}\mathbf{\Psi}(\zeta)e^{\left(\frac{2}{3}\zeta^{3/2} - \frac{i\pi}{6}\right)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix} \ \text{ for } \pi < \arg(\zeta) < \frac{4\pi}{3}.$$

(22)

We'll verify in the lecture that P_1 so defined satisfies the above Riemann–Hilbert problem. The verification is not extremely illuminating: it is straightforward to check that P_1 satisfies the jump relationships on the contours $\arg\zeta = 2\pi/3$ and $\arg\zeta = 4\pi/3$. The verification on the real axis requires the following basic property of the Airy function:

(23)
$$Ai(\zeta) + \omega Ai(\omega\zeta) + \omega^2 Ai(\omega^2\zeta) = 0.$$

The verification of the asymptotic behavior as $\zeta \to \infty$ requires knowledge of the asymptotic behavior of the Airy function as $\zeta \to \infty$:

$$\mathrm{Ai}(\zeta) \sim \frac{1}{2\sqrt{\pi}}\zeta^{-\frac{1}{4}}e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}\sum_{k=0}^{\infty}(-1)^k s_k\left(\frac{2}{3}\zeta^{\frac{3}{2}}\right)^{-k} \quad \text{for } |\arg\zeta| < \pi\,,$$

$$\mathrm{Ai}'(\zeta) \sim -\frac{1}{2\sqrt{\pi}}\zeta^{\frac{1}{4}}e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}\sum_{k=0}^{\infty}(-1)^k t_k\left(\frac{2}{3}\zeta^{\frac{3}{2}}\right)^{-k} \quad \text{for } |\arg\zeta| < \pi\,,$$

$$s_0 = t_0 = 1\,, \quad s_k = \frac{\Gamma(3k+\frac{1}{2})}{54^k k!\,\Gamma(k+\frac{1}{2})}\,, \quad t_k = -\frac{6k+1}{6k-1}s_k\,, \quad \text{for } k \geq 1\,,$$

But what is mysterious from the outset is the arrival on the scene of the infamous Airy function. Whence came this connection to special functions?

As we have seen, there is a general uniqueness theorem for a wide class of matrix Riemann–Hilbert problems, and the Riemann–Hilbert problem we are considering in this section falls within that class. So... we have built *the only* solution to this Riemann–Hilbert problem. Let's understand this connection more completely.

Consider the following *new* matrix valued function:

$$\boldsymbol{\Psi}(\zeta) = P_1(\zeta)e^{-\frac{2}{3}\zeta^{3/2}\sigma_3},\ \zeta \in \mathbb{C}\setminus(-\infty,0]. \tag{24}$$

It is straightforward to verify that $\boldsymbol{\Psi}$ is analytic in the same domain as $P_1(\zeta)$, and moreover, $\boldsymbol{\Psi}(\zeta)$ possesses *constant jumps*! Indeed, the jumps for $\boldsymbol{\Psi}$ are as follows:

$$\boldsymbol{\Psi}_+(\zeta) = \boldsymbol{\Psi}_-(\zeta)w(\zeta), \tag{25}$$

$$w(\zeta) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ for } \zeta \in \mathbb{R}_+, \tag{26}$$

$$w(\zeta) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ for } \arg\zeta = 2\pi/3 \text{ and } \arg\zeta = 4\pi/3, \tag{27}$$

$$w(\zeta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } \zeta \in \mathbb{R}_-. \tag{28}$$

Now in addition to assuming that $\boldsymbol{\Psi}$ exists, let us also assume that it is differentiable in ζ, and that the derivative, $\boldsymbol{\Psi}'(\zeta)$ possesses nice boundary values. It then follows that $\boldsymbol{\Psi}'(\zeta)$ possesses the *same jumps* as $\boldsymbol{\Psi}$ itself. As we have seen before, an immediate consequence is that the ratio, $\boldsymbol{\Psi}'(\zeta)\ \boldsymbol{\Psi}(\zeta)^{-1}$ is *entire*. However, if we in addition assume that the asymptotic behavior of $\boldsymbol{\Psi}'(\zeta)$ may be obtained by differentiating the asymptotic behavior of $\boldsymbol{\Psi}(\zeta)$, we may deduce that the ratio $\boldsymbol{\Psi}'(\zeta)\ \boldsymbol{\Psi}(\zeta)^{-1}$ is not only entire, but also a polynomial in ζ. Skipping the calculations for now (we will do them in class), we find

$$\boldsymbol{\Psi}'(\zeta)\ \boldsymbol{\Psi}(\zeta)^{-1} = \begin{pmatrix} 0 & -1 \\ -\zeta & 0 \end{pmatrix} \tag{29}$$

This is a differential equation, and some further jacking around leads to the Airy equation. We'll see this in class.

References

[1] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, "Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory", *Comm. Pure Appl. Math.*, **52**, 1335–1425, 1999.

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[3] A. Fokas, A. Its, and A. V. Kitaev, "Discrete Painlevé equations and their appearance in quantum gravity", *Commun. Math. Phys.*, **142**, 313–344, 1991.

[4] T. Kriecherbauer and K. T.-R. McLaughlin, Strong Asymptotics of Polynomials Orthogonal with Respect to Freud Weights, *Int. Math. Res. Not.*, No. 6, pp. 299333, 1999.