

LECTURE 27: CLEANING UP THE RIEMANN–HILBERT LABORATORY, AND THEN PROPERTIES OF THE EQUILIBRIUM MEASURE

**Lecture plan.** First we'll verify one of the jump relationships satisfied by the Airy parametrix, and then we will verify the asymptotic behavior of  $P_1$ . Following that we will discuss the transformation *back* from  $\mathbf{E}$  to  $\mathbf{A}$ , and how one obtains asymptotics for the orthogonal polynomials.

BUILDING  $P_1$  VIA AIRY FUNCTIONS

So we consider a full-fledged Riemann–Hilbert problem for  $P_1(\zeta)$ , in the  $\zeta$ -plane.

**Riemann-Hilbert Problem 1.** Find  $P_1(\zeta)$ , satisfying the following three conditions.

**Analyticity.**  $P_1(\zeta)$  is analytic for  $\zeta \in \mathbb{C} \setminus \Sigma_{P_1}$ , and takes continuous boundary values  $(P_1)_+(\zeta)$ ,  $(P_1)_-(\zeta)$  with  $x \in \Sigma$ .

**Jump Condition.** The boundary values are connected by the relation

$$(1) \quad (P_1)_+(\zeta) = (P_1)_-(\zeta) \hat{v}^{(1)}(\zeta), \quad \zeta \in \Sigma_{P_1}$$

**Normalization.** The matrix  $P_1(\zeta)$  is normalized at  $\zeta = \infty$  as follows:

$$P(\zeta) = \zeta^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3} (I + O(\zeta^{-3/2})), \quad \zeta \rightarrow \infty,$$

The jump matrix  $\hat{v}^{(1)}(\zeta)$  is defined as follows:

$$\begin{aligned} \hat{v}^{(1)}(\zeta) &= \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix}, & \arg \zeta = 0, \\ \hat{v}^{(1)}(\zeta) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \arg \zeta = \pi, \\ \hat{v}^{(1)}(\zeta) &= \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, & \arg \zeta = \frac{2\pi}{3}, \\ \hat{v}^{(1)}(\zeta) &= \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, & \arg \zeta = \frac{4\pi}{3}. \end{aligned}$$

We first define the matrix-valued function  $\Psi(\zeta)$  as follows:

$$(2) \quad \Psi(\zeta) = \begin{pmatrix} Ai(\zeta) & Ai(e^{-2i\pi/3}\zeta) \\ Ai'(\zeta) & e^{-2i\pi/3} Ai'(e^{-2i\pi/3}\zeta) \end{pmatrix}, \quad \text{for } 0 < \arg(\zeta) < \pi,$$

$$(3) \quad \Psi(\zeta) = \begin{pmatrix} Ai(\zeta) & -e^{-2i\pi/3} Ai(e^{2i\pi/3}\zeta) \\ Ai'(\zeta) & -Ai'(e^{2i\pi/3}\zeta) \end{pmatrix}, \quad \text{for } -\pi < \arg(\zeta) < 0,$$

where  $\omega = e^{2i\pi/3}$ . Using  $\Psi$ , we now define  $P_1$  as follows:

$$(4) \quad P_1(\zeta) = \sqrt{2\pi} e^{-\frac{i\pi}{12}} \Psi(\zeta) e^{\left(\frac{2}{3}\zeta^{3/2} - \frac{i\pi}{6}\right)\sigma_3} \quad \text{for } |\arg(\zeta)| < \frac{2\pi}{3},$$

$$(5) \quad P_1(\zeta) = \sqrt{2\pi} e^{-\frac{i\pi}{12}} \Psi(\zeta) e^{\left(\frac{2}{3}\zeta^{3/2} - \frac{i\pi}{6}\right)\sigma_3} \begin{pmatrix} 1 & 0 \\ -e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix} \quad \text{for } \frac{2\pi}{3} < \arg(\zeta) < \pi,$$

$$(6) \quad P_1(\zeta) = \sqrt{2\pi} e^{-\frac{i\pi}{12}} \Psi(\zeta) e^{\left(\frac{2}{3}\zeta^{3/2} - \frac{i\pi}{6}\right)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix} \quad \text{for } \pi < \arg(\zeta) < \frac{4\pi}{3}.$$

(7)

We'll verify in the lecture that  $P_1$  so defined satisfies the above Riemann–Hilbert problem. The verification is not extremely illuminating: it is straightforward to check that  $P_1$  satisfies the jump relationships on the contours  $\arg \zeta = 2\pi/3$  and  $\arg \zeta = 4\pi/3$ . The verification on the real axis requires the following basic property of the Airy function:

$$(8) \quad \text{Ai}(\zeta) + \omega \text{Ai}(\omega\zeta) + \omega^2 \text{Ai}(\omega^2\zeta) = 0.$$

Let us consider the jump across  $\mathbb{R}_+$ . To this end, we have the following *alternative formula* for the jump matrix:

$$(9) \quad \hat{v}^{(1)}(\zeta) = (P_1)_-^{-1} (P_1)_+ = e^{-\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \Psi_-(\zeta)^{-1} \Psi_+(\zeta) e^{\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3}.$$

Now to compute the inverse of  $\Psi_-(\zeta)$ , the following fact is useful:

**CLAIM:**  $\det \Psi \equiv \frac{e^{i\pi/6}}{2\pi}$ . (The proof of this is straightforward - you prove that this determinant is constant in  $\zeta$ , then evaluate it as  $\zeta \rightarrow \infty$ .)

Now we may compute the inverse of  $\Psi_-(\zeta)$ , and we find

$$\begin{aligned} \hat{v}^{(1)}(\zeta) &= e^{-\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \times \\ &\quad \frac{2\pi}{e^{i\pi/6}} \begin{pmatrix} -\text{Ai}'(e^{2i\pi/3}\zeta) & e^{-2i\pi/3}\text{Ai}(e^{2i\pi/3}\zeta) \\ -\text{Ai}'(\zeta) & \text{Ai}(\zeta) \end{pmatrix} \begin{pmatrix} \text{Ai}(\zeta) & \text{Ai}(e^{-2i\pi/3}\zeta) \\ \text{Ai}'(\zeta) & e^{-2i\pi/3}\text{Ai}'(e^{-2i\pi/3}\zeta) \end{pmatrix} \times \\ &\quad e^{\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \\ &= e^{-\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \times \\ &\quad \frac{2\pi}{e^{i\pi/6}} \begin{pmatrix} \frac{e^{i\pi/6}}{2\pi} & \omega^4 \text{Ai}(e^{2i\pi/3}\zeta) \text{Ai}'(e^{-2i\pi/3}\zeta) - \text{Ai}'(e^{2i\pi/3}\zeta) \text{Ai}(e^{-2i\pi/3}\zeta) \\ 0 & \frac{e^{i\pi/6}}{2\pi} \end{pmatrix} \times \\ &\quad e^{\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \\ &= e^{-\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \times \\ &\quad \frac{2\pi}{e^{i\pi/6}} \begin{pmatrix} \frac{e^{i\pi/6}}{2\pi} & -\omega^{-2} (\omega^2 \text{Ai}'(e^{2i\pi/3}\zeta) \text{Ai}(e^{-2i\pi/3}\zeta) - \text{Ai}(e^{2i\pi/3}\zeta) \text{Ai}'(e^{-2i\pi/3}\zeta)) \\ 0 & \frac{e^{i\pi/6}}{2\pi} \end{pmatrix} \times \\ &\quad e^{\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \\ &= e^{-\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \frac{2\pi}{e^{i\pi/6}} \begin{pmatrix} \frac{e^{i\pi/6}}{2\pi} & -\omega^{-2} \frac{e^{i\pi/6}}{2\pi} \\ 0 & \frac{e^{i\pi/6}}{2\pi} \end{pmatrix} e^{\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \\ &= e^{-\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \begin{pmatrix} 1 & -\omega^{-2} \\ 0 & 1 \end{pmatrix} e^{\left(\frac{2}{3}\zeta^{3/2} - i\pi/6\right)\sigma_3} \\ &= \begin{pmatrix} 1 & -\omega^2 e^{i\pi/3} e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

The verification of the asymptotic behavior as  $\zeta \rightarrow \infty$  requires knowledge of the asymptotic behavior of the Airy function as  $\zeta \rightarrow \infty$ :

$$\begin{aligned} \text{Ai}(\zeta) &\sim \frac{1}{2\sqrt{\pi}} \zeta^{-\frac{1}{4}} e^{-\frac{2}{3}\zeta^{\frac{3}{2}}} \sum_{k=0}^{\infty} (-1)^k s_k \left(\frac{2}{3}\zeta^{\frac{3}{2}}\right)^{-k} \quad \text{for } |\arg \zeta| < \pi, \\ \text{Ai}'(\zeta) &\sim -\frac{1}{2\sqrt{\pi}} \zeta^{\frac{1}{4}} e^{-\frac{2}{3}\zeta^{\frac{3}{2}}} \sum_{k=0}^{\infty} (-1)^k t_k \left(\frac{2}{3}\zeta^{\frac{3}{2}}\right)^{-k} \quad \text{for } |\arg \zeta| < \pi, \end{aligned}$$

$$s_0 = t_0 = 1, \quad s_k = \frac{\Gamma(3k + \frac{1}{2})}{54^k k! \Gamma(k + \frac{1}{2})}, \quad t_k = -\frac{6k+1}{6k-1} s_k, \quad \text{for } k \geq 1,$$

So, for example, let's pick  $\zeta \rightarrow \infty$ , but restrict that  $\zeta$  remain in the set  $0 < \arg \zeta < 2\pi/3$ . For in that case we may use the following formula for  $P_1(\zeta)$ :

$$(10) \quad P_1(\zeta) = \sqrt{2\pi} e^{-\frac{i\pi}{12}} \begin{pmatrix} Ai(\zeta) & Ai(e^{-2i\pi/3}\zeta) \\ Ai'(\zeta) & e^{-2i\pi/3} Ai'(e^{-2i\pi/3}\zeta) \end{pmatrix} e^{(\frac{2}{3}\zeta^{3/2} - \frac{i\pi}{6})\sigma_3}.$$

Now if  $\zeta$  remains in the set  $0 < \arg \zeta < 2\pi/3$ , then of course  $-2\pi/3 < \arg(e^{-2\pi i/3}\zeta) < 0$ , and so we may use the above asymptotic behavior for both columns. It will be important to use the following basic fact:

**Basic Fact:** If  $\zeta$  remains in the set  $0 < \arg \zeta < 2\pi/3$ , then  $(e^{-2i\pi/3}\zeta)^{3/2} = -\zeta^{3/2}$ .

So, we find

$$\begin{aligned} P_1(\zeta) &= \sqrt{2\pi} e^{-\frac{i\pi}{12}} \begin{pmatrix} \frac{1}{2\sqrt{\pi}\zeta^{1/4}} e^{-\frac{2}{3}\zeta^{3/2}} (1 + \dots) & \frac{1}{2\sqrt{\pi}(e^{-2\pi i/3}\zeta)^{1/4}} e^{\frac{2}{3}\zeta^{3/2}} (1 + \dots) \\ \frac{-\zeta^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}\zeta^{3/2}} (1 + \dots) & \frac{-e^{-2\pi i/3}(e^{-2\pi i/3}\zeta)^{1/4}}{2\sqrt{\pi}} e^{\frac{2}{3}\zeta^{3/2}} (1 + \dots) \end{pmatrix} e^{(\frac{2}{3}\zeta^{3/2} - \frac{i\pi}{6})\sigma_3} \\ &= \frac{e^{-i\pi/12}}{\sqrt{2}} \zeta^{-\sigma_3/4} \begin{pmatrix} (1 + \dots) & e^{i\pi/6} (1 + \dots) \\ -(1 + \dots) & e^{i\pi/6} (1 + \dots) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3} \\ &= \zeta^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ 1- & 1 \end{pmatrix} \left( \mathbb{I} + \mathcal{O}(\zeta^{-3/2}) \right) e^{-\frac{i\pi}{4}\sigma_3}. \end{aligned}$$

#### SUMMARY OF THE ASYMPTOTIC CALCULATION

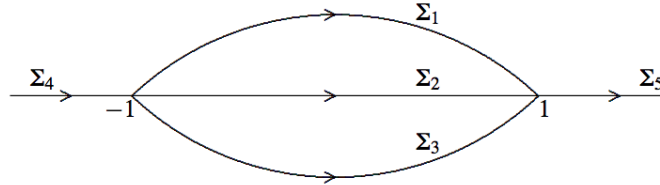
We began with  $\mathbf{A}(z)$  solving the original Riemann–Hilbert problem for orthogonal polynomials. The first transformation was

$$(11) \quad \mathbf{B}(z) := e^{-\frac{N\ell}{2}\sigma_3} \mathbf{A}(z) e^{-N(g(z) - \frac{\ell}{2})\sigma_3}.$$

The second transformation was as follows:

So we then defined  $\mathbf{D}(z)$  as follows:

- For  $z$  outside the “lens shaped region” surrounding the interval  $(-1, 1)$ ,  $\mathbf{D}(z) = B(z)$ .
- For  $z$  within the “upper lens shaped region”, we set  $\mathbf{D}(z) = B(z)v_+(z)^{-1}$ .
- For  $z$  within the “lower lens shaped region”, we set  $\mathbf{D}(z) = B(z)v_-(z)$ .



The matrix  $D$  now solves a new Riemann–Hilbert problem.

**Riemann–Hilbert Problem 2.** Find  $D(z)$ , satisfying the following three conditions.

**Analyticity.**  $\mathbf{D}(z)$  is analytic for  $z \in \mathbb{C} \setminus \Sigma$ , and takes continuous boundary values  $\mathbf{D}_+(z)$ ,  $\mathbf{D}_-(z)$  with  $x \in \Sigma$ .

**Jump Condition.** The boundary values are connected by the relation

$$(12) \quad \mathbf{D}_+(z) = \mathbf{D}_-(z)V_D(z), \quad z \in \Sigma$$

**Normalization.** The matrix  $\mathbf{D}(z)$  is normalized at  $z = \infty$  as follows:

$$(13) \quad \lim_{z \rightarrow \infty} \mathbf{D}(z) = \mathbb{I}.$$

The jump matrix  $V_D$  was defined as follows:

- For  $z \in \Sigma_4 \cup \Sigma_5$ , we have  $V_D(z) = V_B(z)$ .

- For  $z \in \Sigma_2$ , we have

$$(14) \quad V_D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- For  $z \in \Sigma_1$ , we have  $V_D(z) = v_+(z)$ .
- For  $z \in \Sigma_3$ , we have  $V_D(z) = v_-(z)$ .

And it is clear that the new unknown,  $D$ , is analytic of the more complicated union of contours shown above. Moreover, given the above considerations, the jump matrices satisfy the following important property:

**For any  $\delta > 0$ , the jump matrix  $V_D(z)$  is exponentially close to  $\mathbb{I}$  for all values of  $z$  whose distance from  $[-1, 1]$  is greater than  $\delta$ .**

**global approximation to  $D$ .** We then began building a global approximation to  $D$ , by asking if we could find  $\dot{D}$  solving the following *reduced* Riemann–Hilbert problem:

**Riemann-Hilbert Problem 3.** Find  $\dot{D}(z)$  satisfying the following three conditions.

- (1) (*analyticity*) The matrix  $\dot{D}$  is analytic in  $\mathbb{C} \setminus [-1, 1]$ .
- (2) (*Normalization*)  $\dot{D}(z) = \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right)$  as  $z \rightarrow \infty$ .
- (3) (*Boundary values and jump relation*)

$$(15) \quad \dot{D}_+(x) = \dot{D}_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The function  $\dot{D}$  is explicitly known:

$$(16) \quad \dot{D}(z) = \mathbb{F} \begin{pmatrix} a(z) & 0 \\ 0 & \frac{1}{a(z)} \end{pmatrix} \mathbb{F}^{-1}$$

where  $\mathbb{F}$  is the following explicit matrix:

$$(17) \quad \mathbb{F} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

The intuition which we have developed indicates that

$$(18) \quad \mathbf{E}(z) := \mathbf{D}(z) \left( \dot{D}(z) \right)^{-1}$$

should be a new unknown which has *no jump across*  $(-1, 1)$ , and has jumps that are exponentially near to  $\mathbb{I}$  for  $z$  in the contour  $\Sigma$  but bounded away from  $\pm 1$ , and so maybe this quantity satisfies the *guiding principle* we have spoken about in Lecture 21 (and see the discussion in the “RHPSurvey” lecture posted on the website).

But we are not quite home yet: the jump matrices are not uniformly near to  $\mathbb{I}$ . In Lecture 25 we began discussing the local behavior of the jump matrices in a vicinity of  $z = 1$ . The end result was that under the transformation

$$(19) \quad \frac{4}{3}\zeta(z)^{3/2} = 2\pi i N \int_1^x \tilde{\psi}_\beta(s) ds$$

We (more or less) verified that  $\zeta(z)$  is analytic for  $z$  in a neighborhood of  $z = 1$ , and maps a disc of fixed size centered at  $z = 1$  to a neighborhood of  $\zeta = 0$ . The neighborhood in the  $\zeta$ -plane is very large (roughly  $\mathcal{O}(N^{2/3})$ ).

The point to this transformation is that in the new  $\zeta$ -plane, the jump matrices  $V_D(z(\zeta))$  take on the following form:

$$\begin{aligned}\hat{\mathbf{v}}^{(1)}(\zeta) &= \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix}, & \arg \zeta = 0, \\ \hat{\mathbf{v}}^{(1)}(\zeta) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \arg \zeta = \pi, \\ \hat{\mathbf{v}}^{(1)}(\zeta) &= \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, & \arg \zeta = \frac{2\pi}{3}, \\ \hat{\mathbf{v}}^{(1)}(\zeta) &= \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, & \arg \zeta = \frac{4\pi}{3}.\end{aligned}$$

Now the idea here is as follows: *can we construct a matrix valued function which has exactly these jumps in the  $\zeta$  plane? If so, can we port it back over to the  $z$  plane, and use it as a “parametrix” in a neighborhood of  $z = 1$ ?* Let’s call this function  $P_1(\zeta)$ .

**Similar calculations may be carried out in a vicinity of  $z = -1$ , and for those calculations, we will let the analogous function be referred to as  $P_{-1}(\zeta)$ .**

So now we have built three separate matrix valued functions which together may be used as a global approximation to  $\mathbf{D}(z)$ . Here is the definition:

- For all  $z \in \mathbb{C}$  but *outside* discs of radius  $\delta > 0$  centered at  $z = \pm 1$ , we define

$$(20) \quad \mathbf{D}_a(z) = \dot{D}(z).$$

- For  $|z - 1| < \delta$ , we define

$$(21) \quad \mathbf{D}_a(z) = \mathcal{A}_1(z)P_1(\zeta(z)).$$

- For  $|z + 1| < \delta$ , we define

$$(22) \quad \mathbf{D}_a(z) = \mathcal{A}_{-1}(z)P_{-1}(\zeta(z)).$$

**NOTE:** The matrices  $\mathcal{A}_1$  and  $\mathcal{A}_{-1}$  are matrices which are still yet to be determined. They will be *analytic* in a vicinity of 1 and  $-1$ , respectively.

#### HOW GOOD IS THE GLOBAL APPROXIMATION

As we have seen in previous lectures, the only way to assess the approximation  $\mathbf{D}_a$  is to consider the ratio

$$(23) \quad \mathbf{E}(z) := \mathbf{D}(z) \cdot \mathbf{D}_a(z)^{-1},$$

which solves yet another Riemann–Hilbert problem.

**Riemann–Hilbert Problem 4.** *Find  $\mathbf{E}(z)$ , satisfying the following three conditions.*

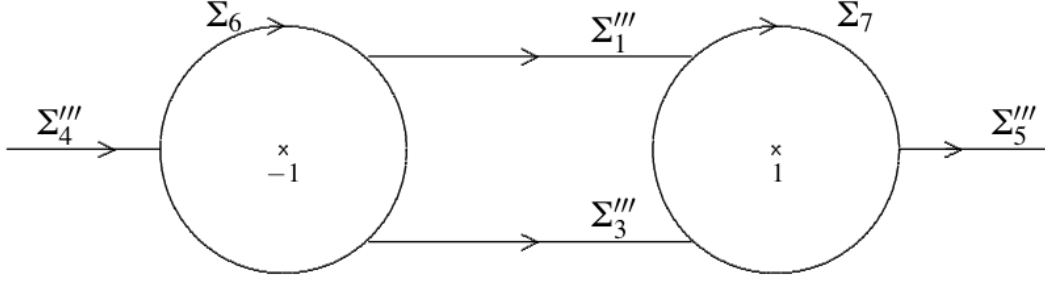
**Analyticity.**  $\mathbf{E}(z)$  is analytic for  $z \in \mathbb{C} \setminus \Sigma_E$ , and takes continuous boundary values  $\mathbf{E}_+(z)$ ,  $\mathbf{E}_-(z)$  with  $x \in \Sigma$ .

**Jump Condition.** *The boundary values are connected by the relation*

$$(24) \quad \mathbf{E}_+(z) = \mathbf{E}_-(z)V_E(z), \quad z \in \Sigma_E$$

**Normalization.** *The matrix  $\mathbf{E}(z)$  is normalized at  $z = \infty$  as follows:*

$$(25) \quad \lim_{z \rightarrow \infty} \mathbf{E}(z) = \mathbb{I}.$$



The jump matrix  $V_E$  is defined as follows:

- For  $z \in \Sigma_4''' \cup \Sigma_5'''$ , we have  $V_E(z) = V_B(z) = \mathbb{I} + \mathcal{O}\left(\left|e^{-2\pi i N \int_1^z \tilde{\psi}_\beta(s) ds}\right|\right)$ .
- For  $z \in \Sigma_1'''$ , we have  $V_E(z) = v_+(z) = \mathbb{I} + \mathcal{O}(e^{-cN})$ .
- For  $z \in \Sigma_3'''$ , we have  $V_E(z) = v_-(z) = \mathbb{I} + \mathcal{O}(e^{-cN})$ .
- For  $z \in \Sigma_7$ , we have  $V_E(z) = \mathcal{A}_1(z)P_1(\zeta(z))\dot{D}(z)^{-1}$ .
- For  $z \in \Sigma_6$ , we have  $V_E(z) = \mathcal{A}_{-1}(z)P_{-1}(\zeta(z))\dot{D}(z)^{-1}$ .

Having built the local parametrices explicitly and verified that they satisfy the jump relations exactly, as well as determined their asymptotic behavior, we now return to the question: *can we determine  $\mathcal{A}_1$  and  $\mathcal{A}_{-1}$  so that the jumps on the contours  $\Sigma_6$  and  $\Sigma_7$  are, finally,  $\mathbb{I} + \mathcal{O}(N^{-1})$ .*

Well, let us consider the jump matrix for  $z \in \Sigma_7$ . Since  $z$  is bounded away from  $z = 1$ , it turns out that  $\zeta$  is  $\mathcal{O}(N^{2/3})$ , and hence we may evaluate the jump relationship  $V_E(z) = \mathcal{A}_1(z)P_1(\zeta(z))\dot{D}(z)^{-1}$  by asymptotically for  $N$  large. The end result of this calculation is:

$$(26) \quad V_E(z) = \mathcal{A}_1(z)\zeta(z)^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ 1- & 1 \end{pmatrix} \left(\mathbb{I} + \mathcal{O}\left(\zeta(z)^{-3/2}\right)\right) e^{-\frac{i\pi}{4}\sigma_3} \mathbb{F} \begin{pmatrix} \frac{1}{a(z)} & 0 \\ 0 & a(z) \end{pmatrix} \mathbb{F}^{-1}.$$

Now, since the above must be  $\mathbb{I} + \mathcal{O}(N^{-1})$ , we require

$$(27) \quad \mathcal{A}_1(z)\zeta(z)^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ 1- & 1 \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3} \mathbb{F} \begin{pmatrix} \frac{1}{a(z)} & 0 \\ 0 & a(z) \end{pmatrix} \mathbb{F}^{-1} \equiv \mathbb{I},$$

which yields

$$(28) \quad \mathcal{A}_1(z) = \left(\zeta(z)^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ 1- & 1 \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3} \mathbb{F} \begin{pmatrix} \frac{1}{a(z)} & 0 \\ 0 & a(z) \end{pmatrix} \mathbb{F}^{-1}\right)^{-1}.$$

Amazingly, you can prove (as a homework assignment) that  $\mathcal{A}_1(z)$  is analytic in a neighborhood of  $z = 1$ . **Hint: prove that  $\mathcal{A}_1(z)$  so defined has no jumps in a vicinity of  $z = 1$ .**

RETURNING TO THE EQUILIBRIUM MEASURE: HOW CAN YOU CALCULATE THIS BEAST?

Well, I didn't have a chance yet to latex this section, so let's see if we get to it in class today.

#### REFERENCES

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