Lecture 27: Cleaning up the Riemann–Hilbert laboratory, and then properties of the equilibrium measure

**Lecture plan.** First we’ll verify one of the jump relationships satisfied by the Airy parametrix, and then we will verify the asymptotic behavior of $P_1$. Following that we will discuss the transformation back from $E$ to $A$, and how one obtains asymptotics for the orthogonal polynomials.

**Building $P_1$ via Airy functions**

So we consider a full-fledged Riemann–Hilbert problem for $P_1(\zeta)$, in the $\zeta$-plane.

**Riemann-Hilbert Problem 1.** Find $P_1(\zeta)$, satisfying the following three conditions.

**Analyticity.** $P_1(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus \Sigma_{P_1}$, and takes continuous boundary values $(P_1)_+(\zeta)$, $(P_1)_-(\zeta)$ with $x \in \Sigma$.

**Jump Condition.** The boundary values are connected by the relation

$$(P_1)_+(\zeta) = (P_1)_-(\zeta) e^{(1)(\zeta)}, \zeta \in \Sigma_{P_1}$$

**Normalization.** The matrix $P_1(\zeta)$ is normalized at $\zeta = \infty$ as follows:

$$P(\zeta) = \zeta^{-\frac{\alpha}{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{\beta}{3}(I + O(\zeta^{-3/2}))}, \zeta \to \infty,$$

The jump matrix $e^{(1)}(\zeta)$ is defined as follows:

$$e^{(1)}(\zeta) = \begin{pmatrix} 1 & e^{-\frac{4}{3} \xi^{3/2}} \\ 0 & 1 \end{pmatrix}, \arg \zeta = 0,$$

$$e^{(1)}(\zeta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \arg \zeta = \pi,$$

$$e^{(1)}(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3} \xi^{3/2}} & 1 \end{pmatrix}, \arg \zeta = \frac{2\pi}{3},$$

$$e^{(1)}(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3} \xi^{3/2}} & 1 \end{pmatrix}, \arg \zeta = \frac{4\pi}{3}.$$

We first define the matrix-valued function $\Psi(\zeta)$ as follows:

$$(2) \quad \Psi(\zeta) = \begin{pmatrix} Ai(\zeta) & Ai(e^{-2\pi/3}) \\ Ai'(\zeta) & e^{-2\pi/3}Ai'(e^{-2\pi/3}) \end{pmatrix}, \text{ for } 0 < \arg(\zeta) < \pi,$$

$$(3) \quad \Psi(\zeta) = \begin{pmatrix} Ai(\zeta) & -e^{-2\pi/3}Ai(e^{2\pi/3}) \\ Ai'(\zeta) & -Ai'(e^{2\pi/3}) \end{pmatrix}, \text{ for } -\pi < \arg(\zeta) < 0,$$

where $\omega = e^{2\pi/3}$. Using $\Psi$, we now define $P_1$ as follows:

$$(4) \quad P_1(\zeta) = \sqrt{2\pi} e^{-\frac{\pi}{3\omega}} \Psi(\zeta) e^{\left(\frac{4}{3} \xi^{3/2} - \frac{\omega}{3}\right)\sigma_3} for |\arg(\zeta)| < \frac{2\pi}{3},$$

$$(5) \quad P_1(\zeta) = \sqrt{2\pi} e^{-\frac{\pi}{3\omega}} \Psi(\zeta) e^{\left(\frac{4}{3} \xi^{3/2} - \frac{\omega}{3}\right)\sigma_3} \begin{pmatrix} 1 & 0 \\ -e^{\frac{4}{3} \xi^{3/2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} for \frac{2\pi}{3} < \arg(\zeta) < \pi,$$

$$(6) \quad P_1(\zeta) = \sqrt{2\pi} e^{-\frac{\pi}{3\omega}} \Psi(\zeta) e^{\left(\frac{4}{3} \xi^{3/2} - \frac{\omega}{3}\right)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3} \xi^{3/2}} & 1 \end{pmatrix} for \pi < \arg(\zeta) < \frac{4\pi}{3},$$

$$(7) \quad \ldots$$
We'll verify in the lecture that $P_1$ so defined satisfies the above Riemann–Hilbert problem. The verification is not extremely illuminating: it is straightforward to check that $P_1$ satisfies the jump relationships on the contours $\arg \zeta = 2\pi/3$ and $\arg \zeta = 4\pi/3$. The verification on the real axis requires the following basic property of the Airy function:

$$A_1(\zeta) + \omega A_1(\omega \zeta) + \omega^2 A_1(\omega^2 \zeta) = 0.$$  

Let us consider the jump across $\mathbb{R}_+$. To this end, we have the following alternative formula for the jump matrix:

$$\hat{v}^{(1)}(\zeta) = (P_1)^{-1}_-(P_1)^+ = e^{-(\frac{2}{3}i\zeta^{3/2}/-i\pi/6)\sigma_3} \Psi_-(\zeta)^{-1} \Psi_+(\zeta) e^{(\frac{2}{3}i\zeta^{3/2}/-i\pi/6)\sigma_3}.$$  

Now to compute the inverse of $\Psi_-(\zeta)$, the following fact is useful:

**CLAIM:** $\det \Psi \equiv \frac{e^{i\pi/6}}{2\pi}$. (The proof of this is straightforward - you prove that this determinant is constant in $\zeta$, then evaluate it as $\zeta \to \infty$.)

Now we may compute the inverse of $\Psi_-(\zeta)$, and we find

$$\hat{v}^{(1)}(\zeta) = e^{-(\frac{2}{3}i\zeta^{3/2}/-i\pi/6)\sigma_3} \times$$

$$\begin{align*}
2\pi & \left( \begin{array}{cc}
-Ai'(e^{2i\pi/3}\zeta) & e^{-2i\pi/3}Ai(e^{2i\pi/3}\zeta) \\
-Ai(\zeta) & Ai(e^{2i\pi/3}\zeta)
\end{array} \right) \\
= & e^{(\frac{2}{3}i\zeta^{3/2}/-i\pi/6)\sigma_3} \times \\
& \frac{2\pi}{e^{i\pi/6}} \left( \begin{array}{cc}
e^{i\pi/6} & \omega^2 Ai(e^{2i\pi/3}\zeta)Ai'(-e^{2i\pi/3}\zeta) - Ai(e^{2i\pi/3}\zeta)Ai'(-e^{2i\pi/3}\zeta) \\
e^{i\pi/6} & e^{i\pi/6}
\end{array} \right) \\
= & e^{-(\frac{2}{3}i\zeta^{3/2}/-i\pi/6)\sigma_3} \times \\
& \frac{2\pi}{e^{i\pi/6}} \left( \begin{array}{cc}
e^{i\pi/6} & -\omega^{-2} (\omega^2 Ai(e^{2i\pi/3}\zeta)Ai(-e^{2i\pi/3}\zeta) - Ai(e^{2i\pi/3}\zeta)Ai'(-e^{2i\pi/3}\zeta)) \\
e^{i\pi/6} & e^{i\pi/6}
\end{array} \right) \\
= & e^{-(\frac{2}{3}i\zeta^{3/2}/-i\pi/6)\sigma_3} \times \\
& \frac{2\pi}{e^{i\pi/6}} \left( \begin{array}{cc}
1 & -\omega^{-2} \\
0 & 1
\end{array} \right) e^{(\frac{2}{3}i\zeta^{3/2}/-i\pi/6)\sigma_3}
\end{align*}$$

The verification of the asymptotic behavior as $\zeta \to \infty$ requires knowledge of the asymptotic behavior of the Airy function as $\zeta \to \infty$:

$$Ai(\zeta) \sim \frac{1}{2\sqrt{\pi}} \zeta^{-\frac{1}{2}} e^{-\frac{2}{3}i\zeta^{3/2}} \sum_{k=0}^{\infty} (-1)^k s_k \left( \frac{2}{3}i\zeta^{3/2} \right)^{-k} \quad \text{for } |\arg \zeta| < \pi,$$

$$Ai'(\zeta) \sim \frac{1}{2\sqrt{\pi}} \zeta^{-\frac{1}{2}} e^{-\frac{2}{3}i\zeta^{3/2}} \sum_{k=0}^{\infty} (-1)^k t_k \left( \frac{2}{3}i\zeta^{3/2} \right)^{-k} \quad \text{for } |\arg \zeta| < \pi,$$

$$s_0 = t_0 = 1, \quad s_k = \frac{\Gamma(3k + \frac{1}{2})}{54^k k! \Gamma(k + \frac{1}{2})}, \quad t_k = -\frac{6k + 1}{6k - 1} s_k, \quad \text{for } k \geq 1.$$
So, for example, let’s pick \( \zeta \to \infty \), but restrict that \( \zeta \) remain in the set \( 0 < \arg \zeta < 2\pi/3 \). For in that case we may use the following formula for \( P_1(\zeta) \):

\[
P_1(\zeta) = \sqrt{2\pi e^{-i\pi/12}} \begin{pmatrix}
\frac{1}{2\sqrt{\pi} e^{\frac{3}{2}i\pi/2}} (1 + \cdots) \\
-\frac{1}{2\sqrt{\pi} e^{-\frac{3}{2}i\pi/2}} (1 + \cdots)
\end{pmatrix} e^{\frac{3}{2}i\pi/4} (1 + \cdots) e^{\frac{3}{2}i\pi/4} (1 + \cdots) e^{\frac{3}{2}i\pi/4} (1 + \cdots)
\]

Now if \( \zeta \) remains in the set \( 0 < \arg \zeta < 2\pi/3 \), then of course \(-2\pi/3 < \arg (e^{-2\pi/3}\zeta) < 0\), and so we may use the above asymptotic behavior for both columns. It will be important to use the following basic fact:

**Basic Fact:** If \( \zeta \) remains in the set \( 0 < \arg \zeta < 2\pi/3 \), then \((e^{-2\pi/3}\zeta)^{3/2} = -\zeta^{3/2}\).

So, we find

\[
P_1(\zeta) = \frac{e^{-i\pi/12}}{\sqrt{2}} \zeta^{-\sigma_3/4} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \left( \mathbb{I} + O\left(\zeta^{-3/2}\right) \right) e^{\frac{i\pi}{6} \sigma_3}.
\]

**SUMMARY OF THE ASYMPIOTIC CALCULATION**

We began with \( A(z) \) solving the original Riemann–Hilbert problem for orthogonal polynomials. The first transformation was

\[
B(z) := e^{-\frac{N}{2} \sigma_3} A(z) e^{-N\left(g(z) - \frac{2}{3}\right)} \sigma_3.
\]

The second transformation was as follows:

So we then defined \( D(z) \) as follows:

- For \( z \) outside the “lens shaped region” surrounding the interval \((-1, 1)\), \( D(z) = B(z) \).
- For \( z \) within the “upper lens shaped region”, we set \( D(z) = B(z) v_+ (z)^{-1} \).
- For \( z \) within the “lower lens shaped region”, we set \( D(z) = B(z) v_-(z) \).

The matrix \( D \) now solves a new Riemann–Hilbert problem.

**Riemann-Hilbert Problem 2.** Find \( D(z) \), satisfying the following three conditions.

**Analyticity.** \( D(z) \) is analytic for \( z \in \mathbb{C} \setminus \Sigma \), and takes continuous boundary values \( D_+(z) \), \( D_-(z) \) with \( x \in \Sigma \).

**Jump Condition.** The boundary values are connected by the relation

\[
D_+(z) = D_-(z) V_D(z) , \ z \in \Sigma.
\]

**Normalization.** The matrix \( D(z) \) is normalized at \( z = \infty \) as follows:

\[
\lim_{z \to \infty} D(z) = \mathbb{I}.
\]

The jump matrix \( V_D \) was defined as follows:

- For \( z \in \Sigma_4 \cup \Sigma_5 \), we have \( V_D(z) = V_B(z) \).
• For \( z \in \Sigma_2 \), we have
\[
V_D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
(14)

• For \( z \in \Sigma_1 \), we have \( V_D(z) = v_+(z) \).

• For \( z \in \Sigma_3 \), we have \( V_D(z) = v_-(z) \).

And it is clear that the new unknown, \( D \), is analytic of the more complicated union of contours shown above. Moreover, given the above considerations, the jump matrices satisfy the following important property:

For any \( \delta > 0 \), the jump matrix \( V_D(z) \) is exponentially close to \( I \) for all values of \( z \) whose distance from \([-1, 1]\) is greater than \( \delta \).

**global approximation to \( D \).** We then began building a global approximation to \( D \), by asking if we could find \( \dot{D} \) solving the following reduced Riemann–Hilbert problem:

**Riemann-Hilbert Problem 3.** Find \( \dot{D}(z) \) satisfying the following three conditions.

1. (analyticity) The matrix \( \dot{D} \) is analytic in \( \mathbb{C} \setminus [-1, 1] \).
2. (Normalization) \( \dot{D}(z) = I + O\left(\frac{1}{z}\right) \) as \( z \to \infty \).
3. (Boundary values and jump relation)
\[
\dot{D}_+(x) = \dot{D}_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
(15)

The function \( \dot{D} \) is explicitly known:
\[
\dot{D}(z) = F \left( \begin{array}{cc} \frac{a(z)}{\alpha(z)} & 0 \\ 0 & \frac{1}{\alpha(z)} \end{array} \right) F^{-1}
\]
(16)

where \( F \) is the following explicit matrix:
\[
F = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.
\]
(17)

The intuition which we have developed indicates that
\[
\dot{D}(z) = D(z) \left( \dot{D}(z) \right)^{-1}
\]
(18)

should be a new unknown which has no jump across \((-1, 1)\), and has jumps that are exponentially near to \( I \) for \( z \) in the contour \( \Sigma \) but bounded away from \( \pm 1 \), and so maybe this quantity satisfies the guiding principle we have spoken about in Lecture 21 (and see the discussion in the “RHPSurvey” lecture posted on the website).

But we are not quite home yet: the jump matrices are not uniformly near to \( I \). In Lecture 25 we began discussing the local behavior of the jump matrices in a vicinity of \( z = 1 \). The end result was that under the transformation
\[
\frac{4}{3} \zeta(z)^{3/2} = 2\pi i N \int_1^z \tilde{\psi}_\beta(s) ds
\]
(19)

We (more or less) verified that \( \zeta(z) \) is analytic for \( z \) in a neighborhood of \( z = 1 \), and maps a disc of fixed size centered at \( z = 1 \) to a neighborhood of \( \zeta = 0 \). The neighborhood in the \( \zeta \)-plane is very large (roughly \( \mathcal{O}(N^{2/3}) \)).

The point to this transformation is that in the new \( \zeta \)-plane, the jump matrices \( V_D(z(\zeta)) \) take on the following form:
Now the idea here is as follows: can we construct a matrix valued function which has exactly these jumps in the ζ plane? If so, can we port it back over to the z plane, and use it as a “parametrix” in a neighborhood of z = 1? Let’s call this function \( P_1(\zeta) \).

Similar calculations may be carried out in a vicinity of \( z = -1 \), and for those calculations, we will let the analogous function be referred to as \( P_{-1}(\zeta) \).

So now we have built three separate matrix valued functions which together may be used as a global approximation to \( D(z) \). Here is the definition:

- For all \( z \in \mathbb{C} \) but outside discs of radius \( \delta > 0 \) centered at \( z = \pm 1 \), we define
  \[
  D_a(z) = \hat{D}(z).
  \]
  (20)

- For \( |z - 1| < \delta \), we define
  \[
  D_a(z) = A_1(z)P_1(\zeta(z)).
  \]
  (21)

- For \( |z + 1| < \delta \), we define
  \[
  D_a(z) = A_{-1}(z)P_{-1}(\zeta(z)).
  \]
  (22)

**NOTE:** The matrices \( A_1 \) and \( A_{-1} \) are matrices which are still yet to be determined. They will be analytic in a vicinity of 1 and \(-1\), respectively.

**HOW GOOD IS THE GLOBAL APPROXIMATION**

As we have seen in previous lectures, the only way to assess the approximation \( D_a \) is to consider the ratio

\[
E(z) := D(z) \cdot D_a(z)^{-1},
\]

which solves yet another Riemann–Hilbert problem.

**Riemann-Hilbert Problem 4.** Find \( E(z) \), satisfying the following three conditions.

- **Analyticity.** \( E(z) \) is analytic for \( z \in \mathbb{C} \setminus \Sigma_E \), and takes continuous boundary values \( E_+(z) \), \( E_-(z) \) with \( x \in \Sigma \).

- **Jump Condition.** The boundary values are connected by the relation
  \[
  E_+(z) = E_-(z)V_E(z), \quad z \in \Sigma_E
  \]
  (24)

- **Normalization.** The matrix \( E(z) \) is normalized at \( z = \infty \) as follows:
  \[
  \lim_{z \to \infty} E(z) = I.
  \]
  (25)
The jump matrix $V_E$ is defined as follows:

- For $z \in \Sigma'' \cup \Sigma'''$, we have $V_E(z) = V_B(z) = I + O\left(\left| e^{-2\pi i N \int_{\tilde{E}} \hat{\nu}_\beta(s) ds} \right| \right)$.
- For $z \in \Sigma''''$, we have $V_E(z) = v_+(z) = I + O\left( e^{-cN} \right)$.
- For $z \in \Sigma'''$, we have $V_E(z) = v_-(z) = I + O\left( e^{-cN} \right)$.
- For $z \in \Sigma_\tau$, we have $V_E(z) = A_1(z) P_1(\zeta(z)) \hat{D}(z)^{-1}$.
- For $z \in \Sigma_0$, we have $V_E(z) = A_{-1}(z) P_{-1}(\zeta(z)) \hat{D}(z)^{-1}$.

Having built the local parametrices explicitly and verified that the satisfy the jump relations exactly, as well as determined their asymptotic behavior, we now return to the question: can we determine $A_1$ and $A_{-1}$ so that the jumps on the contours $\Sigma_6$ and $\Sigma_7$ are, finally, $I + O\left( N^{-1} \right)$.

Well, let us consider the jump matrix for $z \in \Sigma_7$. Since $z$ is bounded away from $z = 1$, it turns out that $\zeta$ is $O\left( N^{2/3} \right)$, and hence we may evaluate the jump relationship $V_E(z) = A_1(z) P_1(\zeta(z)) \hat{D}(z)^{-1}$ by asymptotically for $N$ large. The end result of this calculation is:

$$V_E(z) = A_1(z) \zeta(z)^{-\sigma_3/4} \left( \begin{array}{cc} 1 & \frac{1}{a(z)} \\ 1 & 0 \end{array} \right) \left( I + O\left( \zeta(z)^{-3/2} \right) \right) e^{-\frac{3\pi}{4} \sigma_3} \mathcal{F} \left( \frac{1}{a(z)} \frac{0}{a(z)} \right) F^{-1}.$$

Now, since the above must be $I + O\left( N^{-1} \right)$, we require

$$A_1(z) \zeta(z)^{-\sigma_3/4} \left( \begin{array}{cc} 1 & \frac{1}{a(z)} \\ 1 & 0 \end{array} \right) e^{-\frac{3\pi}{4} \sigma_3} \mathcal{F} \left( \frac{1}{a(z)} \frac{0}{a(z)} \right) F^{-1} \equiv I,$$

which yields

$$A_1(z) = \left( \zeta(z)^{-\sigma_3/4} \left( \begin{array}{cc} 1 & \frac{1}{a(z)} \\ 1 & 0 \end{array} \right) e^{-\frac{3\pi}{4} \sigma_3} \mathcal{F} \left( \frac{1}{a(z)} \frac{0}{a(z)} \right) F^{-1} \right)^{-1}.$$

Amazingly, you can prove (as a homework assignment) that $A_1(z)$ is analytic in a neighborhood of $z = 1$.

**Hint:** prove that $A_1(z)$ so defined has no jumps in a vicinity of $z = 1$.

**Returning to the equilibrium measure:** how can you calculate this beast?

Well, I didn’t have a chance yet to latex this section, so let’s see if we get to it in class today.

**References**


