

**Lecture plan.** For the KdV equation, we will assume that the initial data is such that the associated Lax differential operator possesses one eigenvalue, and we will establish that there emerges a single soliton as  $t$  grows to  $\infty$ .

## RIEMANN–HILBERT PROBLEM FOR KdV

Recall from Lecture 10 that we have shown the following result.

**Theorem.** *Let  $V(x, t)$  evolve according to the KdV equation, with initial data corresponding to the scattering data  $\{r(z), \{E_j, A_j\}_{j=1}^N\}$ . Then the row-vector  $\mathbf{M}(z; x, t)$  is the unique solution of the following Meromorphic Riemann–Hilbert Problem:*

**Problem:** Find  $\mathbf{M}$ , a  $1 \times 2$  matrix, satisfying the following 4 conditions

- (1) Each entry of  $\mathbf{M}$  is a meromorphic function of  $z$ , for  $z \in \mathbb{C}_+ \cup \mathbb{C}_-$ , with continuous boundary values for  $z \in \mathbb{R}$ . These boundary values are denoted  $\mathbf{M}_\pm$ , respectively.
- (2)  $\mathbf{M} = (1, 1) + \mathcal{O}(1/z)$  as  $z \rightarrow \infty$ .
- (3) For each eigenvalue  $E_j$ ,  $\mathbf{M}$  possesses a simple pole at  $z_j^\pm = \pm i\sqrt{-E_j}$ , and

$$(1) \quad \text{res}_{z_j^+} \mathbf{M} = \lim_{z \rightarrow z_j^+} \mathbf{M}(z) \begin{pmatrix} 0 & \frac{e^{2ixz_j^+ + 8it(z_j^+)^3}}{A_j S'_{11}(z_j)} \\ 0 & 0 \end{pmatrix}$$

$$(2) \quad \text{res}_{z_j^-} \mathbf{M} = \lim_{z \rightarrow z_j^-} \mathbf{M}(z) \begin{pmatrix} 0 & 0 \\ \frac{e^{-2ixz_j^- - 8it(z_j^-)^3}}{A_j S'_{11}(z_j)} & 0 \end{pmatrix}$$

- (4) For  $z \in \mathbb{R}$ , the boundary values  $\mathbf{M}_\pm$  are related by a jump relationship:

$$(3) \quad \mathbf{M}_+(z) = \mathbf{M}_-(z) \begin{pmatrix} 1 & -\overline{r(z)} e^{2izx + 8iz^3 t} \\ r(z) e^{-2izx - 8iz^3 t} & 1 - |r(z)|^2 \end{pmatrix}.$$

In addition, the potential  $V(x, t)$  may be extracted from  $\mathbf{M}$  as follows. One computes the behavior of  $\mathbf{M}$  for  $z \rightarrow \infty$ :

$$(4) \quad \mathbf{M}(z; x, t) = (1, 1) + \mathbf{M}^{(1)}(x, t)z^{-1} + \mathcal{O}(z^{-2}), \quad \text{as } z \rightarrow \infty,$$

and then

$$(5) \quad V(x, t) = 2i \frac{\partial}{\partial x} \left( \mathbf{M}^{(1)}(x, t) \right)_1.$$

Here are two natural research projects. The point to the problem is that one can analyze the above Riemann–Hilbert problem just as one is able to analyze the integral which represents the solution to the linearized KdV equation. Take as initial data the *square well*,  $V_0(x) = -b$ , for  $|x| < z$ ,  $V_0(x) = 0$  for  $|x| > b$ . Find the reflection coefficient  $r(z)$ , and choose  $a$  and  $b$  so that there is exactly one eigenvalue  $E_1 < 0$ , and find  $A_1$ . The two projects are to study the behavior of the solution to the KdV equation in two separate limits:

- (1) The small-time behavior of  $V(x, t)$ . For  $t = 0$ , the initial data possesses a jump discontinuity. But for  $t > 0$ , the solution is  $C^\infty$ ! (Prove this.) Now study the behavior of  $V(x, t)$  for  $t \downarrow 0$ . How does a  $C^\infty$  function converge to a function with jump discontinuities? You should see Gibbs' phenomenon!
- (2) The long-time behavior of  $V(x, t)$ . As  $t$  grows, there should emerge from this solution a single soliton, which asymptotically converges to a translate of the soliton solution we found in Lecture 1.

We will now concentrate on the second project, without worrying too much about the exact computation of the reflection coefficient and the eigenvalue. So we will assume that there is exactly one eigenvalue,  $E_1$ , so there are two poles,  $z^\pm$ . We shall further assume that we know  $A_1$ , and  $S'_{11}(z^\pm)$ , and the reflection coefficient  $r(z)$ . The goal is as follows:

**Goal:** Provide an asymptotic description of the solution  $M = M(z; x, t)$  to the above Riemann–Hilbert problem, and from that asymptotic description extract the asymptotic behavior of the solution to the KdV equation.

#### A MODEL ASYMPTOTIC CALCULATION

We will begin the analysis by asking: *what if the reflection coefficient is absent?* In this case, we seek  $M$  which is *meromorphic* in the plane, with residue conditions

$$(6) \quad \text{res}_{z^+} \mathbf{M}_0 = \lim_{z \rightarrow z^+} \mathbf{M}_0(z) \begin{pmatrix} 0 & \frac{e^{2ixz^+ + 8it(z^+)^3}}{AS'_{11}(z^+)} \\ 0 & 0 \end{pmatrix}$$

$$(7) \quad \text{res}_{z^-} \mathbf{M}_0 = \lim_{z \rightarrow z^-} \mathbf{M}_0(z) \begin{pmatrix} 0 & 0 \\ \frac{e^{-2ixz^- - 8it(z^-)^3}}{AS'_{11}(z^-)} & 0 \end{pmatrix}$$

Well, since we have a row-vector with meromorphic entries, with simple poles at  $z^\pm$ , and behaving like (1, 1) as  $z \rightarrow \infty$ , we know that  $M$  must take the form

$$(8) \quad \mathbf{M}_0(z) = \left( \frac{\alpha}{z - z^-} + 1, \frac{\beta}{z - z^+} + 1 \right).$$

Now we ask, how can we determine the constants  $\alpha$  and  $\beta$ ? The residue conditions yield a system of two equations:

$$(9) \quad \begin{pmatrix} 1 & \frac{e^{-2ixz^- - 8it(z^-)^3}}{(z^+ - z^-)AS'_{11}(z^+)} \\ -\frac{e^{2ixz^+ + 8it(z^+)^3}}{(z^+ - z^-)AS'_{11}(z^+)} & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{e^{-2ixz^- - 8it(z^-)^3}}{AS'_{11}(z^-)} \\ -\frac{e^{2ixz^+ + 8it(z^+)^3}}{AS'_{11}(z^+)} \end{pmatrix}$$

So this equation can be solved. From the Theorem, we find that the solution to the KdV equation is  $2i\alpha(x, t)$ , which we will determine in class.

**QUESTION:** How good an approximation is  $\mathbf{M}_0$  to the full solution to the Riemann–Hilbert problem  $\mathbf{M}$ ?

#### A SECOND MODEL PROBLEM

*Next we ask: what happens if there are no poles? The calculations will be carried out in class.*

#### PUTTING IT ALL TOGETHER

*In the end, the new wrinkle is this: what is the proper analog of our Guiding Principle for vector-valued Riemann–Hilbert problems?*