

LECTURE 28: A BASIC UNIVERSALITY RESULT FOR THE KdV EQUATION, COMPLETED

Lecture plan. For the KdV equation, we will assume that the initial data is such that the associated Lax differential operator possesses one eigenvalue, and that the reflection coefficient possesses an analytic extension to a strip around the real axis. We will establish that there emerges a single soliton as t grows to ∞ . In the previous lecture we showed

- If the reflection coefficient is identically zero, and if there is exactly one eigenvalue, then we have a soliton, moving to the right.
- If, on the other hand, there are no eigenvalues, then for $x, t \rightarrow +\infty$ with $\frac{x}{t} \rightarrow \xi$, we have exponential decay of the solution to the KdV equation.

The point to this lecture is to meld these two calculations together, and show that the relative contribution of the reflection coefficient is negligible.

RIEMANN–HILBERT PROBLEM FOR KdV

Recall from Lecture 10 that we have shown the following result.

Theorem. *Let $V(x, t)$ evolve according to the KdV equation, with initial data corresponding to the scattering data $\{r(z), \{E_j, A_j\}_{j=1}^N\}$. Then the row-vector $\mathbf{M}(z; x, t)$ is the unique solution of the following Meromorphic Riemann–Hilbert Problem:*

Problem: Find \mathbf{M} , a 1×2 matrix, satisfying the following 4 conditions

- (1) Each entry of \mathbf{M} is a meromorphic function of z , for $z \in \mathbb{C}_+ \cup \mathbb{C}_-$, with continuous boundary values for $z \in \mathbb{R}$. These boundary values are denoted \mathbf{M}_\pm , respectively.
- (2) $\mathbf{M} = (1, 1) + \mathcal{O}(1/z)$ as $z \rightarrow \infty$.
- (3) For each eigenvalue E_j , \mathbf{M} possesses a simple pole at $z_j^\pm = \pm i\sqrt{-E_j}$, and

$$(1) \quad \text{res}_{z_j^+} \mathbf{M} = \lim_{z \rightarrow z_j^+} \mathbf{M}(z) \begin{pmatrix} 0 & \frac{e^{2ixz_j^+ + 8it(z_j^+)^3}}{A_j S'_{11}(z_j)} \\ 0 & 0 \end{pmatrix}$$

$$(2) \quad \text{res}_{z_j^-} \mathbf{M} = \lim_{z \rightarrow z_j^-} \mathbf{M}(z) \begin{pmatrix} 0 & 0 \\ \frac{e^{-2ixz_j^- - 8it(z_j^-)^3}}{A_j S'_{11}(z_j)} & 0 \end{pmatrix}$$

- (4) For $z \in \mathbb{R}$, the boundary values \mathbf{M}_\pm are related by a jump relationship:

$$(3) \quad \mathbf{M}_+(z) = \mathbf{M}_-(z) \begin{pmatrix} 1 & -\overline{r(z)} e^{2izx + 8iz^3 t} \\ r(z) e^{-2izx - 8iz^3 t} & 1 - |r(z)|^2 \end{pmatrix}.$$

In addition, the potential $V(x, t)$ may be extracted from \mathbf{M} as follows. One computes the behavior of \mathbf{M} for $z \rightarrow \infty$:

$$(4) \quad \mathbf{M}(z; x, t) = (1, 1) + \mathbf{M}^{(1)}(x, t)z^{-1} + \mathcal{O}(z^{-2}), \quad \text{as } z \rightarrow \infty,$$

and then

$$(5) \quad V(x, t) = 2i \frac{\partial}{\partial x} \left(\mathbf{M}^{(1)}(x, t) \right)_1.$$

The goal is as follows:

Goal: *Provide an asymptotic description of the solution $M = M(z; x, t)$ to the above Riemann–Hilbert problem, and from that asymptotic description extract the asymptotic behavior of the solution to the KdV equation.*

A MODEL ASYMPTOTIC CALCULATION

Last time we began the analysis by asking: *what if the reflection coefficient is absent?* In this case, we seek M which is *meromorphic* in the plane, with residue conditions

$$(6) \quad \text{res}_{z^+} \mathbf{M}_0 = \lim_{z \rightarrow z^+} \mathbf{M}_0(z) \begin{pmatrix} 0 & \frac{e^{2ixz^+ + 8it(z^+)^3}}{AS'_{11}(z^+)} \\ 0 & 0 \end{pmatrix}$$

$$(7) \quad \text{res}_{z^-} \mathbf{M}_0 = \lim_{z \rightarrow z^-} \mathbf{M}_0(z) \begin{pmatrix} 0 & 0 \\ \frac{e^{-2ixz^- - 8it(z^-)^3}}{AS'_{11}(z^-)} & 0 \end{pmatrix}$$

Well, since we have a row-vector with meromorphic entries, with simple poles at z^\pm , and behaving like (1, 1) as $z \rightarrow \infty$, we know that M must take the form

$$(8) \quad \mathbf{M}_0(z) = \left(\frac{\alpha}{z - z^-} + 1, \frac{\beta}{z - z^+} + 1 \right).$$

Now we ask, how can we determine the constants α and β ? The residue conditions yield a system of two equations:

$$(9) \quad \begin{pmatrix} 1 & \frac{e^{-2ixz^- - 8it(z^-)^3}}{(z^+ - z^-)AS'_{11}(z^+)} \\ -\frac{e^{2ixz^+ + 8it(z^+)^3}}{(z^+ - z^-)AS'_{11}(z^+)} & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{e^{-2ixz^- - 8it(z^-)^3}}{AS'_{11}(z^+)} \\ -\frac{e^{2ixz^+ + 8it(z^+)^3}}{AS'_{11}(z^+)} \end{pmatrix}$$

So this equation can be solved. From the Theorem, we find that the solution to the KdV equation is $2i\partial_x \alpha(x, t)$.

Remark: In class, we assumed that the residue condition took the form

$$(10) \quad \text{res}_{z^+} \mathbf{M}_0 = \lim_{z \rightarrow z^+} \mathbf{M}_0(z) \begin{pmatrix} 0 & e^{-2x+8t} \\ 0 & 0 \end{pmatrix}$$

$$(11) \quad \text{res}_{z^-} \mathbf{M}_0 = \lim_{z \rightarrow z^-} \mathbf{M}_0(z) \begin{pmatrix} 0 & 0 \\ e^{-2x+8t} & 0 \end{pmatrix}$$

However, this is inconsistent, since upon double checking, it turns out to be the case that $AS'_{11}(z^\pm)$ is purely imaginary, whereas in class we assumed this quantity was equal to $+1$, which is certainly not imaginary. We can correct this by replacing e^{-2x+8t} by ie^{-2x+8t} . Today we will assume the *generic situation*.

QUESTION: *How good an approximation is \mathbf{M}_0 to the full solution to the Riemann–Hilbert problem \mathbf{M} ?*

A SECOND MODEL PROBLEM

Next we ask: *what happens if there are no poles?* The calculations we carried out in class are as follows:

- Factor the jump matrix as follows:

$$(12) \quad \begin{pmatrix} 1 & -\overline{r(z)}e^{2izx+8iz^3t} \\ r(z)e^{-2izx-8iz^3t} & 1 - |r(z)|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r(z)e^{-2izx-8iz^3t} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\overline{r(z)}e^{2izx+8iz^3t} \\ 0 & 1 \end{pmatrix}$$

- Define $\mathbf{N}(z)$ in four regions: two strips, one above the real axis and one below the real axis, and the rest of the plane, which is then two different half-planes:

for $z = x + iy$ with $0 < y < \delta$, define

$$(13) \quad \mathbf{N}(z) = \mathbf{M}(z) \begin{pmatrix} 1 & \overline{r(z)}e^{2izx+8iz^3t} \\ 0 & 1 \end{pmatrix},$$

for $z = x + iy$ with $-\delta < y < 0$, define

$$(14) \quad \mathbf{N}(z) = \mathbf{M}(z) \begin{pmatrix} 1 & 0 \\ r(z)e^{-2izx-8iz^3t} & 1 \end{pmatrix},$$

for $z = x + iy$ with $\delta < y$, define

$$(15) \quad \mathbf{N}(z) = \mathbf{M}(z),$$

for $z = x + iy$ with $y < -\delta$, define

$$(16) \quad \mathbf{N}(z) = \mathbf{M}(z).$$

We verified that the jumps for the new matrix unknown $\mathbf{N}(z)$ are uniformly $\mathbb{I} + \mathcal{O}(e^{-c|t|})$, provided $x, t \rightarrow \infty$, with $\frac{x}{t} \rightarrow \xi > 0$.

PUTTING IT ALL TOGETHER

In the end, the new wrinkle is this: what is the proper analog of our *Guiding Principle* for vector-valued Riemann–Hilbert problems?

Our calculation is the following: we let $x + 4(z^+)^2 t = \zeta$, which is assumed to remain bounded as $t \rightarrow \infty$. Thus $\xi = -4(z^+)^2 > 0$. The steps are as follows:

- Build a full matrix analogue of \mathbf{M}_0 , called $\tilde{\mathbf{M}}_0$, by enforcing two different asymptotic conditions as $z \rightarrow \infty$:

$$(17) \quad \tilde{\mathbf{M}}_0 \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{as } z \rightarrow \infty.$$

- From \mathbf{M} , define \mathbf{N} but ensure that the strips do not contain the poles.
- Now consider a full matrix RHP, for $\tilde{\mathbf{N}}$, with the asymptotics as $z \rightarrow \infty$ given by

$$(18) \quad \tilde{\mathbf{N}}(z) \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{as } z \rightarrow \infty.$$

and residue conditions given by the same conditions as for \mathbf{M} .

- Remove the poles from this extended problem by considering

$$(19) \quad \mathbf{E}(z) = \tilde{\mathbf{N}}\tilde{\mathbf{M}}_0^{-1}.$$

- The new unknown \mathbf{E} is a matrix valued function which solves a RHP, for which the jump matrices are uniformly $\mathbb{I} + \mathcal{O}(e^{-c|t|})$, and so our standard guiding principle applies.
- Finally, one unravels the asymptotic calculations.