LECTURE 1: BASICS OF NONLINEAR WAVES

The wave equation. Consider the wave equation

$$u_{tt} - u_{xx} = 0\,,$$

with u, x, and t real. Typically pose an *initial-value problem* to find u given $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, and subject to suitable boundary conditions on a finite or infinite interval of x. General solution is *d'Alembert's formula*:

$$u(x,t) = F_+(x-t) + F_-(x+t)\,,$$

with $F_\pm$ arbitrary. Determined from initial data by

$$f(x) = F_+(x) + F_-(x)\,, \qquad g(x) = -F'_+(x) + F'_-(x)\,.$$

Thus,

$$F'_\pm(x) = \frac{1}{2}f'(x) \mp \frac{1}{2}g(x)\,.$$

Integrate and use $f(x) = F_+(x) + F_-(x)$:

$$F_\pm(x) = \frac{1}{2}f(x) \mp \frac{1}{2}\int_y^x g(\xi)\,d\xi\,,$$

where y is arbitrary (its effect cancels out of $u(x,t)$). Notes:

- $F_+(x-t)$ and $F_+(x-t)$ represent *traveling wave solutions* of the wave equation. They propagate without change of form at constant speed. If F_+ and F_- represent isolated wave forms (say have compact support), The two wave forms will eventually separate from one another and thus a complicated initial condition can resolve asymptotically into a sum of simple waves.
- Of course for all time $u(x,t)$ is a sum of simple waves. This reflects the *superposition principle* shared by all linear equations: whenever $u_1(x,t)$ and $u_2(x,t)$ are two solutions, then so is the sum $u(x,t) = u_1(x,t) + u_2(x,t)$.

The dynamics of the wave equation implied by d'Alembert's formula and the principle of superposition can be explored using the Mathematica notebook `Solitons.nb`.

The linear Klein-Gordon equation. Dispersion. A simple modification of the wave equation is the following equation

$$u_{tt} - u_{xx} + u = 0\,.$$

This is called a (linear) Klein-Gordon equation. It is also a linear equation, and so has a superposition principle. However, there are no localized traveling wave solutions. How do we know this? Substitute $u(x,t) = F(\xi)$ with $\xi = x - ct$ and by the chain rule obtain

$$u_{xx} = F''(\xi)\,, \qquad u_{tt} = c^2F''(\xi)\,,$$

so under the substitution the equation becomes

$$F''(\xi) + \frac{1}{c^2-1}F(\xi) = 0\,,$$

so if $c^2 < 1$ then

$$F(\xi) = a_+e^{\xi/\sqrt{1-c^2}} + a_-e^{-\xi/\sqrt{1-c^2}}\,,$$

which is unbounded and not localized, and if $c^2 > 1$ then

$$F(\xi) = a\cos(\xi/\sqrt{c^2-1}) + b\sin(\xi/\sqrt{c^2-1})\,,$$

which is bounded but periodic (not pulse-like).

The nature of these sinusoidal traveling wave solutions explains why this equation doesn't support localized traveling waves. Generally a *wavetrain* solution of a wave equation is one of the form

$$u(x,t) = Ae^{i(kx-\omega t)}$$

and A is the *amplitude*, k is the *wavenumber*, and ω is the *frequency.* These are not independent. Upon substitution into the linear Klein-Gordon equation we see that this is a solution as long as

$$-\omega^2 + k^2 + 1 = 0\,.$$

This formula is called a *dispersion relation.* Since the *phase velocity* of the wave train is $v_p = \omega/k$, we see that in this problem

$$v_p^2 = 1 + \frac{1}{k^2}\,,$$

which depends nontrivially on k. Thus waves of different lengths travel with different speeds. This phenomenon is known as *dispersion.* Since by Fourier theory we can write the general solution as

$$u(x,t) = \int_{-\infty}^{\infty} \left[A_+(k)e^{i(kx-\omega_+(k)t)} + A_-(k)e^{i(kx-\omega_-(k)t)}\right]\,dk$$

a general solution is made up of waves traveling at different speeds with respect to each other. This ultimately leads to the distortion of any wave form that does not resemble one of the basic Fourier components.

It should also be noted that Fourier theory gives a solution algorithm for the linear Klein-Gordon equation, even though there is no d'Alembert formula. That is, the functions $A_\pm(k)$ are determined in terms of $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ by Fourier transforms. Thus the Fourier transform and inverse transform give a commutative diagram for solving the problem (draw it).

The dynamics of the linear Klein-Gordon equation, and in particular, the effects of dispersion, can be explored using the Mathematica notebook `Solitons.nb`.

A nonlinear Klein-Gordon equation. Solitary waves. If we want to further modify the equation to bring coherent structures back into the picture, we have to add nonlinearity. Consider the equation

$$u_{tt} - u_{xx} + \frac{4}{\pi}\left(\sin(2\pi u) + \frac{1}{3}\sin(6\pi u)\right) = 0\,.$$

The nonlinear terms here are chosen to be odd and periodic but are otherwise arbitrary (they happen to be the first two terms of the Fourier sine series of 1 on $0 < u < 1/2$). This equation has interesting localized traveling waves. With $u(x,t) = F(\xi)$ and $\xi = x - ct$, we arrive at

$$F''(\xi) + \frac{4}{\pi(c^2-1)}\left(\sin(2\pi F) + \frac{1}{3}\sin(6\pi F)\right) = 0\,.$$

This is a nonlinear ODE, but it is easy to analyze by looking at the phase portrait in the (F, F') plane. Multiplying by F' we find that

$$\frac{d}{d\xi}\left[\frac{1}{2}\left(\frac{dF}{d\xi}\right)^2 - \frac{4}{\pi(c^2-1)}\left(\frac{1}{2\pi}\cos(2\pi F) + \frac{1}{18\pi}\cos(6\pi F)\right)\right] = 0$$

so the solution curves in the phase plane are level curves of an energy. If $c^2 < 1$ we have saddle points at $F = n$, $n \in \mathbb{Z}$ (and $F' = 0$) and center points halfway inbetween. If $c^2 > 1$ it's the other way around. Three kinds of "motions" in ξ:

- Periodic motion. Libration of the pendulum.
- Unbounded (in F) motion. Rotation of the pendulum.
- Separatrices. Heteroclinic orbits connecting saddle points.

The latter correspond to localized solutions of the nonlinear Klein-Gordon equation! Suppose $c^2 < 1$. To find the localized traveling waves it is enough to look at the energy level through $F = F' = 0$:

$$\frac{1}{2}\left(\frac{dF}{d\xi}\right)^2 - \frac{4}{\pi(c^2-1)}\left(\frac{1}{2\pi}\cos(2\pi F) + \frac{1}{18\pi}\cos(6\pi F)\right) = 0\,.$$

This first-order equation may be solved directly for ξ in terms of F because

$$\frac{d\xi}{dF} = \pm\sqrt{\frac{1-c^2}{2}} \cdot \frac{1}{\sqrt{\int_0^F \Phi(g)\,dg}}\,,$$

where

$$\Phi(u) := \frac{4}{\pi}\left(\sin(2\pi u) + \frac{1}{3}\sin(6\pi u)\right)\,.$$

As Φ is positive for $0 < u < 1$, $d\xi/dF$ is always nonzero, so by the implicit function theorem we get $F = F(\xi)$. (Every monotone function has an inverse). Therefore, $F(\xi)$ has the shape of a "front". There are traveling wave fronts of all speeds with $c^2 < 1$. This is different from the linear wave equation which only admitted the speeds $c = \pm 1$. Note the relativistic interpretation of the fronts: the characteristic width of the traveling wave solution $u(x,t) = F(x - ct)$ is proportional to $\sqrt{1 - c^2}$. The faster it goes, the shorter it is. The scaling is exactly that of special relativity. Also, because the passage of the front increases the angle $2\pi u$ by 2π, there is a kind of twisting going on, which motivates the terminology of calling these traveling waves *kinks*.

Localized traveling waves in nonlinear wave equations are called *solitary waves* to contrast them with solutions that are periodic in ξ (and therefore resemble a whole train of waves instead of just one). Once dispersion is in the picture, nonlinearity is essential to have solitary waves in a system. The propagation of the solitary wave should be thought of as a dynamical balance between dispersive "forces" that try to pull the wave apart, and nonlinear "forces" that try to compress it together. You can think of a solitary wave in terms of a group of kids of different sizes all walking along together. If they are walking on the pavement, then some kids walk faster than others and eventually the group spreads out, which is like dispersion. Now put the same group of kids on a huge trampoline, and the ones who get out in front suffer the disadvantage of having to walk uphill while the ones who fall behind are given a boost by walking downhill. The more kids are present, the greater the effect is, which is the essential property of nonlinearity. Combining the effects of nonlinearity and dispersion, the group of kids walking on the trampoline just remains the same size — a solitary wave. This analogy was told to me by Al Scott.

The fact that we are now looking at a nonlinear equation means that we can no longer count on a superposition principle. Sums of solutions are no longer solutions. Also, there is no longer an algorithm for solving initial-value problems. That's just how it is.

The solitary wave solutions of the nonlinear Klein-Gordon equation, and the nonlinear dynamics that occur when they interact in absence of any superposition principle, can be explored using the Mathematica notebook `Solitons.nb`.

The sine-Gordon equation. Solitons. Another example of a nonlinear Klein-Gordon equation is just

$$u_{tt} - u_{xx} + \sin(2\pi u) = 0\,,$$

which is somewhat comically called the sine-Gordon equation. Exactly the same kind of reasoning as before, now using

$$\Phi(u) = \sin(2\pi u)$$

gives a family of solitary wave solutions (kinks) parametrized by velocities $c^2 < 1$. In this case, we can find the traveling waves explicitly:

$$F(\xi) = \frac{2}{\pi}\tan^{-1}\left(\exp\left(\pm\sqrt{\frac{2\pi}{1-c^2}}(\xi - \xi_0)\right)\right)\,.$$

Now, we can again carry out similar numerical experiments to examine collisions of these solitary waves.

You can explore the interactions of solitary waves in the sine-Gordon equation using the Mathematica notebook `Solitons.nb`.

Notes:

- The interaction is now "clean". There is no radiation shed.
- There is a "phase shift": after the collision the kinks reemerge unscathed except that they are shifted somewhat from where they would have been had there been no interaction.

This strange behavior of the solitary waves in the sine-Gordon equation justifies our promoting them to have a new name: *solitons*. Since the dynamics allows any number of solitons to propagate and "pass through each other" (albeit with a phase shift) the velocities of the solitons are observable *constants of the motion*. Since an initial condition could in principle be rigged to contain an arbitrary number of solitons, there are evidently an arbitrary number of conserved quantities for this equation. In the theory of mechanics a Hamiltonian system with a sufficient number of conserved quantities (in involution with respect to each other) makes the mechanical system integrable by quadratures (you can find so-called action-angle variables). The sine-Gordon equation is an example of an *infinite-dimensional integrable system*.

There is also evidently a kind of "nonlinear superposition principle" for the sine-Gordon equation. In one form it is the following: write the sine-Gordon equation in characteristic form $r = x + t$, $s = x - t$ as

$$u_{rs} = \sin(u)\,.$$

and consider the relations relating two functions u and v:

$$\frac{1}{2}(u+v)_r = a\sin\left(\frac{u-v}{2}\right)\,, \qquad \frac{1}{2}(u-v)_s = \frac{1}{a}\sin\left(\frac{u+v}{2}\right)\,.$$

By cross-differentiation, it follows that both

$$u_{rs} = \sin(u)\,, \qquad v_{rs} = \sin(v)$$

so if u is a solution and we determine v through the first-order equations above, we get another solution of the same equation!

Finally, there is a “nonlinear analogue of the Fourier transform”. The inverse-scattering transform. This is going to be a main topic of the course.