# Lecture 2: The Korteweg-de Vries Equation

This is a lecture about some of the properties of the Korteweg-de Vries equation, and its role in the history of the subject of soliton theory.

**The KdV equation. Symmetry properties.** An equation for $u(x,t)$ is said to be a Korteweg-de Vries (KdV) equation (Korteweg and de Vries, 1895) if it is of the form

$$u_t + \alpha u u_x + \beta u_{xxx} = 0\,,$$

for some constants $\alpha$ and $\beta$. The coefficients are somewhat arbitrary, in the sense that we can change them into others, or get rid of them altogether (that is, make them all equal to one), by choosing appropriate units for the equation. The first thing we can try to do is to choose a new spatial scale, so that for some constant $A$ we have

$$x = AX$$

defining some new length scale $X$ that goes through one unit when $x$ goes through $A$ units. Similarly we can pick some constant $B$ and set

$$t = BT$$

defining some new time scale $T$. Then, using the chain rule

$$\frac{\partial}{\partial x} = \frac{dX}{dx} \cdot \frac{\partial}{\partial X} = \frac{1}{A}\frac{\partial}{\partial X}$$

and

$$\frac{\partial}{\partial t} = \frac{dT}{dt} \cdot \frac{\partial}{\partial T} = \frac{1}{B}\frac{\partial}{\partial T}$$

so the equation becomes

$$\frac{1}{B}u_T + \frac{\alpha}{A}u u_X + \frac{\beta}{A^3}u_{XXX} = 0\,,$$

or after multiplying through by $B$,

$$u_T + \frac{\alpha B}{A}u u_X + \frac{\beta B}{A^3}u_{XXX} = 0\,.$$

Now, the KdV equation is a nonlinear equation, and as such it is *not* left unchanged by scaling of $u$. Therefore, we may define a further rescaling by choosing $C$ to be some unit for $u$, and then setting

$$u = CU$$

defining some new dependent variable $U(X,T)$. The differential equation then becomes

$$U_T + \frac{\alpha BC}{A}U U_X + \frac{\beta B}{A^3}U_{XXX} = 0\,.$$

Now if we like we can choose all coefficients to be 1 by choosing values of $A$, $B$, and $C$ so that

$$\frac{\alpha BC}{A} = 1 \qquad \text{and} \qquad \frac{\beta B}{A^3} = 1\,.$$

For example,

$$A = 1\,, \qquad B = \frac{1}{\beta}\,, \qquad C = \frac{\beta}{\alpha}$$

does the trick. But we may also notice that we had more unknowns to determine than we had equations to determine them, which is evidence of a *symmetry group* of the equation. That is, for any $\lambda \neq 0$, the choice

$$A = \lambda\,, \qquad B = \lambda^3\,, \qquad C = \lambda^{-2}$$

leaves the KdV equation invariant. This implies that whenever $u(x,t)$ is a solution of

$$u_t + \alpha u u_x + \beta u_{xxx} = 0\,,$$

then so is $\lambda^2 u(\lambda x, \lambda^3 t)$ for any constant $\lambda \neq 0$. For obvious reasons this kind of symmetry group is called a *scaling symmetry group* or sometimes a *similarity transformation group*. Note that contained in this group is the discrete transformation group associated with $\lambda = \pm 1$. This amounts to the observation that whenever $u(x,t)$ is a solution, so is $u(-x,-t)$.

Another kind of symmetry group of the KdV equation involves going into a frame of reference moving with constant velocity $c$ with respect to the fixed frame. This means that instead of using the independent variables $x$ and $t$ we want to use the independent variables

$$\xi = x - ct\,, \qquad \tau = t\,.$$

Then, since by the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\cdot\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial x}\cdot\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \xi}\,,$$

and

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\cdot\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t}\cdot\frac{\partial}{\partial \tau} = -c\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau}\,,$$

the KdV equation becomes

$$u_\tau + (\alpha u - c)\,u_\xi + \beta u_{\xi\xi\xi} = 0\,.$$

This has not left the equation invariant. But now we may notice that if we add a constant to $u$:

$$u = U + \frac{c}{\alpha}$$

then we arrive at

$$U_\tau + \alpha U U_\xi + \beta U_{\xi\xi\xi} = 0\,.$$

Therefore, whenever $u(x,t)$ is a solution of

$$u_t + \alpha u u_x + \beta u_{xxx} = 0\,,$$

then so is $u(x - ct, t) + c/\alpha$. This kind of symmetry group is called a *translational symmetry group* or a *Galilean transformation group*.

**Origin of the KdV equation. The period 1834–1895.** In the early days of mathematical modeling of water waves, it was assumed that the wave height was small compared to the water depth, which leads to linear dispersive equations a representative model of which is

$$u_t + u_{xxx} = 0\,.$$

Such equations are somewhat satisfying in this regard because they have solutions that resemble waves traveling along with constant speed and fixed profile along the water surface, just like one sees in nature. To find them, we go into a moving frame with speed $c$ by introducing new variables $\xi = x - ct$ and $\tau = t$, and then we seek solutions of the resulting equation that are independent of $\tau$:

$$u = f(\xi) \qquad \text{implies} \qquad -cf'(\xi) + f'''(\xi) = 0\,.$$

Integrating once,

$$f''(\xi) - cf(\xi) = B$$

where $B$ is an integration constant. If $c = -k^2 < 0$, then $f$ will be bounded as a function of $\xi$, and we get

$$f(\xi) = A\cos(k(\xi - \xi_0)) + \frac{B}{k^2} = A\cos(k(\xi - \xi_0)) + D\,,$$

where $D$ is another arbitrary constant. Thus, the family of bounded traveling wave solutions to this linear equation is exhausted by periodic sinusoidal wave shapes. The speed of propagation depends on the wavenumber $k$, since $c = -k^2$. (Note that these waves all propagate to the left only.) There is alot that can be built out of these waves by superposition (it is a linear equation) but the basic point of view at this time in history was that the only waves that travel along at a constant speed without changing their form were periodic trains of waves.

In 1834 an event occurred that began to change this point of view. John Scott-Russell's account of his accidental observation of what he called a "great wave of translation" and what turned out to be the solitary wave solution of the KdV equation is best described by his own words, written to the British Association in 1844 as part of his "Report on Waves":

> I believe I shall best introduce the phaenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

You should go back and look again at the picture of Scott-Russell's "great wave of translation" as it was recreated in 1995 at the naming ceremony of the Scott-Russell aqueduct on the Union canal connecting Glasgow and Edinburgh where Scott-Russell had made his original observation.

The water wave theory needed to be corrected to be able to reproduce the solitary wave of Scott-Russell, and the essential modification required was to throw out the assumption that the wave height was small compared with the water depth (it was still assumed that the wave was long compared with the depth, however). This modification makes the equations of motion *nonlinear.* In 1895, Korteweg and de Vries obtained as a model of water waves in this situation the equation that bears their name:

$$u_t + uu_x + u_{xxx} = 0\,.$$

We are assuming that we have used scaling to make the coefficients all one. This equation has within it the solitary wave of Scott-Russell. Again we look for traveling waves $u = f(\xi = x - ct)$:

$$-cf'(\xi) + f(\xi)f'(\xi) + f'''(\xi) = 0\,.$$

Noting that $ff' = (f^2)'/2$, we can integrate one time to get

$$-cf(\xi) + \frac{1}{2}f(\xi)^2 + f''(\xi) = A$$

where $A$ is an integration constant. Writing this in the form

$$f'' + \left[\frac{1}{2}f^2 - cf - A\right] = 0$$

we may view this as a nonlinear oscillator equation with a nonlinear "restoring force". The oscillator is at equilibrium when the velocity is zero ($f' = 0$) and when the force is zero. The latter condition can be satisfied if $A > -c^2/2$; otherwise there are no equilibria. Note that once we are in such a situation, without loss of generality we may choose $A = 0$ since we may add a constant to $f$ at the cost of adjusting the wave-speed $c$ according to the Galilean symmetry group. If we do this, then there are two equilibria for $f$: $f = 0$ and $f = 2c$. Assuming $c > 0$, we have a saddle point at $f = 0$ and a center point at $f = 2c$.

You should sketch the phase portrait showing the trajectories of this equation in the $(f, f')$ plane.

To have a solitary wave, we want a solution that tends to constant states as $\xi \to \pm\infty$. Here we see that the only choice is the homoclinic orbit connecting the saddle point with itself. To find it, we integrate once more by multiplying by $f'$:

$$\frac{1}{2}(f')^2 + \frac{1}{6}f^3 - \frac{c}{2}f^2 = E$$

where $E$ is an integration constant. For the level curve through the saddle, we have $f = f' = 0$ so $E = 0$. Noting that there is no harm in writing $f = g^2$ for the (positive) homoclinic orbit, this becomes

$$2g^2(g')^2 + \frac{1}{6}g^6 - \frac{c}{2}g^4 = 0$$

or

$$(g')^2 + \frac{1}{12}g^4 - \frac{c}{4}g^2 = 0$$

which is the homoclinic orbit equation for the Duffing oscillator, and can be solved in view of the identities:

$$\mathrm{sech}^2(x) + \tanh^2(x) = 1\,, \qquad \frac{d}{dx}\mathrm{sech}(x) = -\mathrm{sech}(x)\tanh(x)\,.$$

Thus, assuming $g(\xi) = a\text{sech}(b(\xi - \xi_0))$, and substituting,

$$a^2b^2\text{sech}^2(b(\xi - \xi_0))\tanh^2(b(\xi - \xi_0)) + \frac{1}{12}a^4\text{sech}^4(b(\xi - \xi_0)) - \frac{c}{4}a^2\text{sech}^2(b(\xi - \xi_0)) = 0$$

or

$$b^2\tanh^2(b(\xi - \xi_0)) + \frac{a^2}{12}\text{sech}^2(b(\xi - \xi_0)) - \frac{c}{4} = 0\,,$$

Therefore if $b^2 = a^2/12 = c/4$ then we have found a solution. The solitary wave of Scott-Russell is

$$u(x,t) = 3c\,\text{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - x_0)\right)$$

for any wave speed $c > 0$. Note that while the linear waves (bounded traveling wave solutions of $u_t + u_{xxx} = 0$) travel to the left only, the solitary waves (localized traveling wave solutions of $u_t + uu_x + u_{xxx} = 0$) travel to the right only.

The solitary wave can exist in the KdV equation because this equation contains effects of both nonlinearity and dispersion, and these effects are dynamically balanced by the solitary wave solution. We recall once again the analogy of the kids walking on the trampoline.