

Mathematical structure of the KdV equation (continuation).

Infinite-dimensional Hamiltonian systems. In continuum-mechanical problems like those of wave propagation, we are concerned with the way that a function u of a continuous variable x evolves in time t . The Hamiltonian formalism generalizes easily to this context. The phase space now consists of functions $u(x)$ instead of finite-dimensional vectors. Functions on the phase space become *functionals* $F[u]$ returning a numerical value for each input function $u(\cdot)$. We restrict attention to functionals having the form

$$F[u] = \int_{-\infty}^{\infty} f(u, u_x, u_{xx}, \dots) dx.$$

The component index j of the vector \vec{u} with components u_j gets generalized to the continuous variable x , and in place of the gradient $\nabla_u F$ of a function we have the *variational derivative* of a functional written $\delta F/\delta u$ and calculated as

$$F[u + \epsilon v] = F[u] + \epsilon \int_{-\infty}^{\infty} \frac{\delta F}{\delta u}(x) v(x) dx + O(\epsilon^2).$$

For example, if

$$F[u] := \int_{-\infty}^{\infty} [u(x)^3 + u(x)u_{xx}(x)] dx,$$

then

$$\frac{\delta F}{\delta u}(x) = 3u(x)^2 + 2u_{xx}(x),$$

(integrating by parts twice for one of the terms to replace v_{xx} by v). Finally, the matrix $\mathbf{J}(u)$ gets generalized to a linear operator \mathcal{J} on the space of functions $u(x)$ such that the skew-symmetry property and Jacobi identity hold for the corresponding Poisson bracket, which is developed with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x) dx.$$

So, for example, the KdV equation is a Hamiltonian system in which

$$\mathcal{J} := \frac{\partial}{\partial x},$$

That is, the KdV equation can be written in the form

$$u_t = \mathcal{J} \frac{\delta H}{\delta u} = \left(\frac{\delta H}{\delta u} \right)_x.$$

What is the Hamiltonian functional that works here? We need to find a functional H so that

$$\frac{\delta H}{\delta u} = -\frac{1}{2}u^2 - u_{xx}.$$

From our previous example, it is easy to see that we may take

$$H[u] := - \int_{-\infty}^{\infty} \left[\frac{1}{6}u(x)^3 + \frac{1}{2}u(x)u_{xx}(x) \right] dx$$

Note that this gives us a third conserved local density (after u and u^2) since H is automatically a conserved quantity and it is of the form

$$H[u] = \int_{-\infty}^{\infty} D(u, u_x, \dots) dx \quad \text{with} \quad D(u, u_x, \dots) := -\frac{1}{6}u^3 - \frac{1}{2}uu_{xx}.$$

Note that for the KdV equation, the Poisson bracket takes the form

$$\{F, G\} := \int_{-\infty}^{\infty} \frac{\delta F}{\delta u}(x) \frac{d}{dx} \frac{\delta G}{\delta u}(x) dx.$$

Could the KdV equation be a completely integrable system? The fact that the heights of the solitons in Zabusky and Kruskal's simulations were preserved throughout the collisions suggested to them that there were many conserved quantities in the KdV equation. Indeed, from looking at the numerical simulations of KdV one gets the idea that the heights of the various solitons could be harvested as conserved quantities because:

- They can be measured from an initial condition (at least one in which the solitons are very far from one another).
- They are apparently constants of the motion (at least if one doesn't try to measure them during the complicated nonlinear interactions).

Since in principle an initial condition could be rigged that contains an arbitrary number of well-separated solitons, it starts to appear that maybe the KdV equation supports an infinite number of conserved quantities.

This led to a sudden, intense interest in the local conservation laws consistent with the KdV equation. By brute force, several more than the first three we have already seen were obtained (in other words, just by clever guessing). In fact, Zabusky and Kruskal themselves had found a few more, and later Robert Miura was able to find a total of ten local conservation laws consistent with KdV. Of course this is not enough to prove anything, but soon a general method was discovered, which now goes by the name of the *Gardner transform*:

$$u = w + \epsilon w_x + A\epsilon^2 w^2,$$

where ϵ and A are arbitrary parameters. Note that $\epsilon = 0$ gives the identity transformation. We want to substitute this into KdV

$$u_t + uu_x + u_{xxx} = 0$$

and find out what equation w satisfies. So, by direct calculation,

$$u_t = w_t + \epsilon w_{xt} + 2A\epsilon^2 w w_t = \left[1 + \epsilon \frac{\partial}{\partial x} + 2A\epsilon^2 w \right] w_t,$$

and

$$\begin{aligned} uu_x + u_{xxx} &= ww_x + w_{xxx} + \epsilon (ww_{xx} + w_x^2 + w_{xxx}) + \epsilon^2 (3Aw^2 w_x + 2Aw w_{xx} + (1 + 6A)w_x w_{xx}) \\ &\quad + \epsilon^3 (2Aw w_x^2 + Aw^2 w_{xx}) + 2A^2 \epsilon^4 w^3 w_x \\ &= \left[1 + \epsilon \frac{\partial}{\partial x} + 2A\epsilon^2 w \right] [ww_x + w_{xxx} + A\epsilon^2 w^2 w_x] + \epsilon^2 (1 + 6A)w_x w_{xx}. \end{aligned}$$

Now pick $A = -1/6$. Then

$$u_t - uu_x - u_{xxx} = \left[1 + \epsilon \frac{\partial}{\partial x} - \frac{\epsilon^2}{3} w \right] \left[w_t - ww_x - w_{xxx} + \frac{\epsilon^2}{6} w^2 w_x \right].$$

Hence, if w satisfies the *Gardner equation*

$$w_t + ww_x - \frac{\epsilon^2}{6} w^2 w_x + w_{xxx} = 0,$$

then $u = w + \epsilon w_x - \epsilon^2 w^2/6$ satisfies KdV (but not necessarily vice-versa). Note that the Gardner equation can also be written as a local conservation law:

$$w_t + \left[\frac{1}{2} w^2 - \frac{\epsilon^2}{18} w^3 + w_{xx} \right]_x = 0.$$

But furthermore, the Gardner transform may be inverted to express w in terms of u , at least at the level of a formal series:

$$w \sim \sum_{n=0}^{\infty} \epsilon^n w_n.$$

By comparing powers of ϵ in the Gardner transform, the coefficients w_n evidently have to satisfy

$$w_0 = u, \quad w_1 = -w_{0,x} = -u_x, \quad w_2 = -w_{1,x} + \frac{1}{6} w_0^2 = u_{xx} + \frac{1}{6} u^2,$$

and for $n \geq 3$,

$$w_n = -w_{n-1,x} + \frac{1}{3}uw_{n-2} + \frac{1}{6} \sum_{k=1}^{n-3} w_k w_{n-2-k}.$$

Substituting the formal expansion of w in powers of ϵ into the Gardner equation, we see by collecting powers of ϵ that what we really have is a *generating function for local conservation laws*. Indeed the generating function for conserved local densities is w itself:

$$D(\epsilon) = w \sim \sum_{n=0}^{\infty} \epsilon^n D_n$$

with

$$D_0 = u, \quad D_1 = -u_x, \quad D_2 = u_{xx} + \frac{1}{6}u^2,$$

and so on. The generating function for the corresponding fluxes is

$$F(\epsilon) = \frac{1}{2}w^2 - \frac{\epsilon^2}{18}w^3 + w_{xx} \sim \sum_{n=0}^{\infty} \epsilon^n F_n$$

with

$$F_0 = \frac{1}{2}w_0^2 + w_{0,xx} = \frac{1}{2}u^2 + u_{xx}, \quad F_1 = w_0 w_1 + w_{1,xx} = -uu_x - u_{xxx},$$

and so on. Clearly, an arbitrary number of local conservation laws can be generated in this way. They are independent, in the sense that at each level, higher and higher powers of u are introduced. However, only half of them are nontrivial; note that whenever one has a local conservation law of the form

$$D_t + F_x = 0,$$

where additionally it turns out that D has the form

$$D = \frac{\partial E}{\partial x}$$

where E is a polynomial in u and its x derivatives, then the fundamental theorem of calculus gives

$$\frac{d}{dt} \int_{-\infty}^{\infty} D dx = 0,$$

but furthermore the left hand side is also zero. A nontrivial local conservation law is therefore one for which the conserved local density D is not a perfect derivative.

In any case, there is now clearly an infinite number of independent local conservation laws. The question now arises: is this enough to solve the KdV equation?

The breakthrough. The inverse-scattering transform. The mathematical breakthrough came in 1967, in work of Gardner, Greene, Kruskal, and Miura. The existence of an infinite number of local conservation laws for KdV suggested that it might be an infinite-dimensional analogue of a completely integrable Hamiltonian system. This would mean that the KdV equation could somehow be solved, in an analogue of obtaining and solving the elementary equations for the angular variables $(\varphi_1, \dots, \varphi_N)$ on the torus $M(\vec{f})$.

There is another nonlinear equation, very similar in appearance to KdV, that can be solved exactly (this had been known for at least 50 years or so):

$$u_t + uu_x - 3u_{xx} = 0$$

which is called *Burgers' equation*. (The coefficient of 3 is not important; it can be scaled out; but we keep it to make a formal connection with KdV below.) It is not Hamiltonian. Indeed, the u_{xx} term represents diffusion rather than dispersion. This nonlinear equation can be solved because it can be linearized through the *Cole-Hopf transform*: one introduces a new dependent variable ψ related to u by the formula

$$u = -6 \frac{\psi_x}{\psi}.$$

Substituting this into Burgers' equation one finds that a solution of Burgers' equation is obtained whenever ψ satisfies

$$\psi_t - 3\psi_{xx} = 0$$

This is just the heat (or diffusion) equation. It is linear, and can be solved explicitly by Fourier transforms. The Cole-Hopf transform therefore *linearizes* Burgers' equation.

The story is told that Gardner, Greene, Kruskal, and Miura were looking for a generalization of the Cole-Hopf transform that might be applicable to the KdV equation. They were working on the premise that since KdV involves more derivatives, then so should the transform. So they tried

$$u = -6 \frac{\psi_{xx}}{\psi}.$$

This did not work. So they thought to take advantage of the Galilean symmetry group of the KdV equation which allows one to add a constant to u at the cost of going into a moving frame of reference. Therefore they considered instead

$$u + E = -6 \frac{\psi_{xx}}{\psi}.$$

Then they started to think that maybe they should view this as an equation to be solved for ψ given u instead of the other way around. Thus they wrote on the blackboard

$$-6\psi_{xx} - u\psi = E\psi.$$

They were staring at this equation on the blackboard, when somebody walked by in the corridor and said: "Hey, are you guys doing quantum mechanics in there?" Of course this equation was just Schrödinger's equation from quantum mechanics describing the wavefunction ψ corresponding to a particle of energy E in moving in the potential $V = -u$ (with $\hbar^2/(2m) = 6$).

Suppose that V decays rapidly to zero as $|x| \rightarrow \infty$. Then the *spectrum* of Schrödinger's equation is (by one definition) the collection of values of E in the complex plane for which there is a solution $\psi(x)$ that is bounded uniformly for all x . Moreover, we divide the spectrum qualitatively into two parts:

- The *discrete spectrum* or *point spectrum* consists of values E for which there exists a solution $\psi(x)$ that beyond being bounded in x actually decays to zero as $|x| \rightarrow \infty$ in such a sense that $\psi \in L^2(\mathbb{R})$, that is, $|\psi|^2$ is integrable over \mathbb{R} . For suitable $V(x)$ the discrete spectrum for the Schrödinger operator consists of a finite number N of negative real values of E : $E = -\kappa_n^2$, $n = 1, 2, \dots, N$. The values of E in the point spectrum are the *eigenvalues* of the Schrödinger operator with potential V . If $E = -\kappa_n^2$ is an eigenvalue, and the corresponding $L^2(\mathbb{R})$ solution is called $\psi_n(x)$ (an *eigenfunction*) then we may assume $\psi_n(x)$ is a real-valued function, and then of course $K\psi_n(x)$ is another eigenfunction for any real K (Schrödinger's equation is linear in ψ), so by choosing K properly we may assume that ψ_n is *normalized*, i.e.

$$\int_{-\infty}^{\infty} \psi_n(x)^2 dx = 1.$$

The normalized eigenfunction has asymptotic behavior for large positive x given by

$$\psi_n(x) = c_n e^{-\kappa_n x / \sqrt{6}} (1 + o(1)),$$

as $x \rightarrow +\infty$. The constant c_n is determined up to a sign, and is called the *norming constant* associated with the eigenfunction ψ_n or the eigenvalue $E = -\kappa_n^2$.

- The *continuous spectrum* is (for our purposes) the rest of the spectrum. For suitable $V(x)$ decaying to zero with large x , the continuous spectrum for the Schrödinger operator consists of all nonnegative real $E = k^2$, $k \in \mathbb{R}$. For each $k \in \mathbb{R}$, there is a solution of the Schrödinger equation that has the asymptotic form

$$\psi(x; k) = e^{ikx/\sqrt{6}} + r(k)e^{-ikx/\sqrt{6}} + o(1)$$

as $x \rightarrow -\infty$ and

$$\psi(x; k) = t(k)e^{ikx/\sqrt{6}} + o(1)$$

as $x \rightarrow +\infty$. Interpreting $e^{ikx/\sqrt{6}}$ as a wave propagating to the right and $e^{-ikx/\sqrt{6}}$ as a wave propagating to the left, it is natural to call $r(k)$ the *reflection coefficient* and $t(k)$ the *transmission coefficient* for the Schrödinger operator with potential V .

This gave the four an idea they didn't have before. Why not calculate the eigenvalues E assuming that the potential $V = -u$ depends not only on x but on some external parameter t , such that $u(x, t)$ solves the KdV equation? The idea is that for each fixed t one has a potential function V and tries to find the eigenvalues E , which in general depend on t through the parametric dependence in V . When $u(x, t)$ changes according to KdV, the eigenvalues should change in some fashion, and the aim was to find out how.

Here is the miracle they found: the eigenvalues did not change at all. As long as the t dependence of $u(x, t)$ is obtained from the KdV equation, the eigenvalues E do not depend on t . Considered as a functional of u , each eigenvalue $E = -\kappa_n^2$ is a constant of motion of the KdV equation! Moreover, they used the KdV equation to calculate how the associated norming constants c_n would vary as the parameter t is changed. They found

$$c_n(t) = c_n(0)e^{A\kappa_n^3 t}$$

for some overall real constant A . Thus the norming constants *did* depend on t , but in a very simple and explicit way.

Excited by this result, they also calculated the time dependence of the reflection coefficient $r(k)$. They found

$$r(k; t) = r(k; 0)e^{iBk^3 t}$$

for some overall real constant B . In particular, this means that $|r(k; t)|$ is independent of t for all real k . For each real k the modulus of the reflection coefficient, viewed as a functional of u , is a constant of motion of the KdV equation! The time dependence of $r(k; t)$, is, however, completely simple and explicit.

The linear phases $Bk^3 t$ and $A\kappa_n^3 t$ should be viewed as being in complete analogy with the linear flow of the variables φ_n on the invariant simultaneous level set of all of the constants of motion of a completely integrable system. The machinery is all here now: an arbitrary number of conserved functionals (eigenvalues $E = -\kappa_j^2$ and $|r(k)|$, viewed as functionals of u) and corresponding linear motion of phases (logarithms of norming constants and $\arg(r(k))$). What we have learned is that all of these objects have an additional *spectral* interpretation in terms of the Schrödinger operator with potential $V = -u$.

The only thing missing is an answer to the following question: if the eigenvalues $E = \kappa_n^2$ and corresponding norming constants $c_n(t)$ are known along with the reflection coefficient $r(k; t)$ at some time $t > 0$ according to the above formulae (which only require that we can calculate them for $t = 0$, that is, from the given initial condition $u(x, 0)$ for the KdV equation), can we somehow get back from them the corresponding potential $V = -u(x, t)$? If so, then we have found the solution of the initial-value problem for the KdV equation. The answer is affirmative, as the *spectral data* we know turns out to be exactly what is required to solve the problem of *inverse scattering*. That is, there is a procedure for determining the potential V of a Schrödinger operator given the discrete spectrum, norming constants, and reflection coefficient. This had been known in the physics literature since the 1950's.

These are then the steps for solving the KdV equation:

- (1) Given the initial condition $u(x, 0)$, consider $V = -u$ as a potential function in Schrödinger's equation, and calculate the discrete spectrum $E = -\kappa_n^2$, associated norming constants $c_n = c_n(0)$, and reflection coefficient $r(k) = r(k; 0)$.
- (2) Introduce explicitly dependence of these objects on t : the eigenvalues $E = -\kappa_n^2$ remain fixed, while $c_n(t) = c_n(0)e^{A\kappa_n^3 t}$ and $r(k; t) = r(k; 0)e^{iBk^3 t}$.
- (3) Carry out the procedure of inverse-scattering to determine $V = -u(x, t)$ from this "time-evolved" spectral data.

Step 1 is sometimes called the *direct scattering problem* and step 3 is the *inverse-scattering problem*. The whole method is called the *inverse-scattering transform* by direct analogy with the Fourier transform for solving linear problems.

Note that it turns out that there is no "change of variables" formula like the Cole-Hopf transform for linearizing the KdV equation. The KdV equation can indeed be linearized, but it is with the help of the spectral theory of the Schrödinger operator, and not by means of an explicit change of variables formula.