LECTURE 5: UNIVERSAL NATURE OF SOME INTEGRABLE EQUATIONS

Universal nature of the KdV equation. Far from being tied to one application, the KdV equation turns out to pop up under very general conditions in mathematical modeling. This makes it a *universal* model for certain kinds of nonlinear waves, and strengthens the importance of its solvability via the inverse-scattering transform. We will illustrate this through two examples.

The Fermi-Pasta-Ulam lattice. Recall the FPU lattice model

$$m\frac{d^2q_n}{dt^2} = V'(q_{n+1}-q_n) - V'(q_n - q_{n-1})\,,$$

where q_n is the displacement from equilibrium of the nth particle of mass m, and $V(\Delta)$ is the potential energy of the springs connecting the particles. Suppose the amplitude of the displacements is small: $q_n = \epsilon Q_n$ for some small parameter $\epsilon > 0$. Then, we may Taylor expand to get

$$\begin{aligned} m\frac{d^2Q_n}{dt^2} &= V''(0)\left[Q_{n+1} - 2Q_n + Q_{n-1}\right] \\ &\quad + \epsilon\frac{V'''(0)}{2}\left[(Q_{n+1}-Q_n)^2 - (Q_n - Q_{n-1})^2\right] \\ &\quad + \epsilon^2\frac{V^{(IV)}(0)}{6}\left[(Q_{n+1}-Q_n)^3 - (Q_n - Q_{n-1})^3\right] + O(\epsilon^3)\,. \end{aligned}$$

This shows that when ϵ is small we have a small correction to a linear equation for Q_n, which means we are in a *weakly nonlinear* setting. The linear terms correspond to $\epsilon = 0$; the problem with $\epsilon = 0$ has solutions of the form

$$Q_n(t) = Ae^{i(\delta n - \omega t)}$$

for some amplitude A, (discrete) wavenumber δ, and frequency ω. To have a solution of the linearized problem imposes a relation between δ and ω, the *dispersion relation* for this problem:

$$\omega^2 = \frac{2V''(0)}{m}[1-\cos(\delta)]\,.$$

If we plot the two branches of ω as a function of δ, we see that they cross at $\delta = 0$, and moreover that $\delta = 0$ is a point of inflection for the two curves: $\omega''(0) = 0$. If we consider solutions that are superpositions of elementary traveling waves near $\delta = 0$ in this situation, and turn on the effect of the perturbing terms ($\epsilon > 0$ but small) we will obtain the KdV equation.

To localize in the Fourier domain near $\delta = 0$ is the same thing as introducing a slowly varying scale in place of n. Thus we set $X = \epsilon n$ and $T = \epsilon t$, and we view Q_n as a sampled value of a smooth function of X:

$$Q_n(t) = Q(X, T; \epsilon)\,.$$

It follows by Taylor expansion of Q that

$$Q_{n\pm1} = Q \pm \epsilon Q_X + \frac{\epsilon^2}{2}Q_{XX} \pm \frac{\epsilon^3}{6}Q_{XXX} + \frac{\epsilon^4}{24}Q_{XXXX} + O(\epsilon^5)\,,$$

and by the chain rule $d^2Q_n/dt^2 = \epsilon^2 Q_{TT}$. The system of ODEs for $Q_n(t)$ therefore becomes

$$mQ_{TT} = V''(0)Q_{XX} + \epsilon^2\left[\frac{V''(0)}{12}Q_{XXXX} + V'''(0)Q_XQ_{XX}\right] + O(\epsilon^3)\,.$$

If we drop the $O(\epsilon^3)$, this is equivalent to a scaled version of the Boussinesq equation of the first homework set via the substitution $u = Q_X$.

To finish the derivation, we separate out the waves moving in two different directions by going into a frame of reference moving with the right-going waves:

$$\xi = X - \sqrt{\frac{V''(0)}{m}}T = X - \omega'(0)T\,, \qquad \tau = \epsilon^2 T\,,$$

and the equation of motion becomes

$$-2\sqrt{mV''(0)}Q_{\xi\tau} = \frac{V''(0)}{12}Q_{\xi\xi\xi\xi} + V'''(0)Q_\xi Q_{\xi\xi} + O(\epsilon)\,.$$

Dropping the $O(\epsilon)$ term and setting $F = Q_\xi$ gives a Korteweg-de Vries equation:

$$2\sqrt{mV''(0)}F_\tau + V'''(0)FF_\xi + \frac{V''(0)}{12}F_{\xi\xi\xi} = 0\,.$$

Water waves. The model equation for an irrotational, inviscid fluid moving in a plane perpendicular to the surface of the earth with horizontal coordinate x and vertical coordinate y is just Laplace's equation for a velocity potential ϕ:

$$\phi_{xx} + \phi_{yy} = 0\,, \qquad \text{for} \qquad -h_0 < y < h(x,t)$$

with boundary conditions at the bottom

$$\phi_y = 0\,, \qquad \text{at} \qquad y = -h_0\,,$$

and at the water/air interface

$$h_t + \phi_x h_x = \phi_y\,, \qquad \text{and} \qquad \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gh = 0\,, \qquad \text{at} \qquad y = h(x,t)\,.$$

Here g is the acceleration due to gravity, $h(x,t)$ is the curve giving the free boundary, and h_0 is the undisturbed depth of the fluid. Nondimensionalizing by the length scale h_0 and the time scale $\sqrt{h_0/g}$ amounts to rewriting the equations with $h_0 = 1$ and $g = 1$.

For small disturbances the simplest approximation is just to linearize the (nondimensionalized) equations:

$$\phi_{xx} + \phi_{yy} = 0\,, \qquad \text{for} \qquad -1 < y < h(x,t)$$

with boundary conditions at the bottom

$$\phi_y = 0\,, \qquad \text{at} \qquad y = -1\,,$$

and at the water/air interface

$$h_t = \phi_y\,, \qquad \text{and} \qquad \phi_t + h = 0\,, \qquad \text{at} \qquad y = h(x,t)\,.$$

For any real k, Laplace's equation has solutions of the form

$$\phi = e^{ikx}\left[A(t)\cosh(k(y+1)) + B(t)\sinh(k(y+1))\right]$$

and to satisfy the boundary condition at $y = -1$ we must have $B(t) = 0$. So,

$$\phi_y = kA(t)e^{ikx}\sinh(k(y+1))\,, \qquad \text{and} \qquad \phi_t = A'(t)e^{ikx}\cosh(k(y+1))\,.$$

These are approximately $kA(t)e^{ikx}\sinh(k)$ and $A'(t)e^{ikx}\cosh(k)$ respectively when $y = h(x,t)$ and the interface is close to $y = 0$. Eliminating h in the boundary conditions at the free surface then gives

$$kA(t)e^{ikx}\sinh(k) = h_t = -\phi_{tt} = -A''(t)e^{ikx}\cosh(k)\,,$$

or

$$A''(t) + k\tanh(k)A(t) = 0\,.$$

Therefore $A(t)$ is a linear combination of $e^{\pm i\omega t}$ with

$$\omega^2 = k\tanh(k)\,.$$

This is the dispersion relation for water waves. Note as in the FPU problem that two branches cross at $k = 0$ and that this is the only point of inflection for the branches $\omega = \omega(k)$. This means that we are going to find the KdV equation upon expanding about this linearized situation.

To start, we can try to solve the Laplace equation using power series about $y = -1$ (the Cauchy-Kovaleskaya method). Assume that

$$\phi(x,y,t) = \phi_0(x,t) + \sum_{n=2}^{\infty}\phi_n(x,t)(y+1)^n\,.$$

The linear term is missing to satisfy $\phi_y = 0$ at $y = -1$. Substituting into $\phi_{xx} + \phi_{yy} = 0$ gives recurrence relations for the coefficients that are directly solved in terms of $\phi_0(x,t)$:

$$\phi_n(x,t) = 0\,, \qquad \text{for } n \text{ odd}$$

and

$$\phi_{2k}(x,t) = \frac{(-1)^k}{(2k)!}\frac{\partial^{2k}\phi_0}{\partial x^{2k}}(x,t)\,.$$

To localize about $k=0$ amounts to looking for solutions that depend slowly on x (and then, by the dispersion relation, on t). Thus we suppose that

$$\phi_0(x,t) = w(X,T)$$

where the appropriate scaling in this case is $X=\epsilon^{1/2}x$ and $T=\epsilon^{1/2}t$ for some small parameter ϵ. This gives us

$$\phi(x,y,t) = w(X,T) - \frac{\epsilon}{2}(y+1)^2 w_{XX} + \frac{\epsilon^2}{24}(y+1)^4 w_{XXXX} + O(\epsilon^3)\,.$$

We also make the problem weakly nonlinear by scaling the dependent variables:

$$h(x,t) = \epsilon G(X,T)\,, \qquad \text{and} \qquad w(X,T) = \epsilon^{1/2}N(X,T)\,.$$

All that remains is to satisfy the boundary conditions at $y=h=\epsilon G$, which now take the form

$$G_T + (y+1)N_{XX} + \epsilon\left[N_X G_X - \frac{1}{6}(y+1)^3 N_{XXXX}\right] = O(\epsilon^2)\,, \qquad \text{at } y=\epsilon G$$

and

$$N_T + G - \frac{\epsilon}{2}(y+1)^2 N_{XXT} = O(\epsilon^2)\,, \qquad \text{at } y=\epsilon G\,.$$

Expanding about $y=0$ the free boundary conditions become

$$G_T + N_{XX} + \epsilon\left[N_X G_X + G N_{XX} - \frac{1}{6}N_{XXXX}\right] = O(\epsilon^2)\,,$$

and

$$N_T + G - \frac{\epsilon}{2}N_{XXT} = O(\epsilon^2)\,.$$

Eliminating G gives a closed equation for N:

$$N_{TT} - N_{XX} + \frac{\epsilon}{6}N_{XXXX} - \frac{\epsilon}{2}N_{XXTT} + \epsilon N_T N_{XX} + \epsilon N_X N_{XT} = O(\epsilon^2)\,.$$

This equation also says that $N_{TT} = N_{XX} + O(\epsilon)$, so the fourth derivatives may be combined:

$$N_{TT} - N_{XX} - \frac{\epsilon}{3}N_{XXXX} + \epsilon N_T N_{XX} + \epsilon N_X N_{XT} = O(\epsilon^2)\,.$$

The leading (linear) terms would suggest motion to the left or right with speed 1, so let's go into a right-moving frame:

$$\xi = X - T\,, \qquad \tau = \epsilon T\,,$$

which turns our equation into

$$2N_{\xi\tau} + \frac{1}{3}N_{\xi\xi\xi\xi} + 2N_\xi N_{\xi\xi} = O(\epsilon)\,,$$

or, dropping the $O(\epsilon)$ and setting $F = N_X$, we arrive at the KdV equation:

$$F_\tau + FF_\xi + \frac{1}{6}F_{\xi\xi\xi} = 0\,.$$

Another universal equation: the nonlinear Schrödinger equation. An even more ubiquitous equation than KdV is the nonlinear Schrödinger (NLS) equation

$$iA_t + \frac{\omega''(k)}{2}A_{xx} + \beta|A|^2 A = 0\,.$$

If $\beta\omega''(k) > 0$ this is called a *focusing* NLS equation and otherwise it is a *defocusing* NLS equation. It is, in both cases, another integrable equation.

Derivation of focusing NLS from the mKdV equation. The *modified* KdV equation is

$$\varphi_t + \varphi^2\varphi_x + \varphi_{xxx} = 0\,.$$

The scaling $\varphi = \epsilon u$ makes this equation weakly nonlinear when ϵ is small:

$$u_t + \epsilon^2 u^2 u_x + u_{xxx} = 0\,.$$

Introduce multiple spatial scales $X_0 = x$ and $X_1 = \epsilon x$, and multiple time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$. Then if we write $u(x,t)$ in the form

$$u(x,t) = U(X_0, X_1, T_0, T_1, T_2)$$

by the chain rule the function U must satisfy

$$\frac{\partial U}{\partial T_0} + \epsilon\frac{\partial U}{\partial T_1} + \epsilon^2\frac{\partial U}{\partial T_2} + \epsilon^2 U^2\frac{\partial U}{\partial X_0} + \epsilon^3 U^2\frac{\partial U}{\partial X_1} + \frac{\partial^3 U}{\partial X_0^3} + 3\epsilon\frac{\partial^3 U}{\partial X_0^2\partial X_1} + 3\epsilon^2\frac{\partial^3 U}{\partial X_0\partial X_1^2} + \epsilon^3\frac{\partial^3 U}{\partial X_1^3} = 0\,.$$

We propose to determine U perturbatively, via an expansion of the form

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + O(\epsilon^3)$$

as $\epsilon \to 0$. Substituting and collecting powers, we get

$$\frac{\partial U_0}{\partial T_0} + \frac{\partial^3 U_0}{\partial X_0^3} = 0\,,$$

$$\frac{\partial U_1}{\partial T_0} + \frac{\partial^3 U_1}{\partial X_0^3} = -\frac{\partial U_0}{\partial T_1} - 3\frac{\partial^3 U_0}{\partial X_0^2\partial X_1}\,,$$

and

$$\frac{\partial U_2}{\partial T_0} + \frac{\partial^3 U_2}{\partial X_0^3} = -\frac{\partial U_1}{\partial T_1} - 3\frac{\partial^3 U_1}{\partial X_0^2\partial X_1} - \frac{\partial U_0}{\partial T_2} - U_0^2\frac{\partial U_0}{\partial X_0} - 3\frac{\partial^3 U_0}{\partial X_0\partial X_1^2}\,.$$

As a solution of the equation for U_0, we take for some real $k \neq 0$,

$$U_0 = Ae^{i(kX_0 + k^3 T_0)} + \circledast\,, \qquad A = A(X_1, T_1, T_2)\,.$$

The equation governing U_1 then becomes

$$\frac{\partial U_1}{\partial T_0} + \frac{\partial^3 U_1}{\partial X_0^3} = -\left[\frac{\partial A}{\partial T_1} - 3k^2\frac{\partial A}{\partial X_1}\right]e^{i(kX_0 + k^3 T_0)} + \circledast\,.$$

Now since the forcing term is, as a function of X_0 and T_0, a solution of the homogeneous problem, we will have resonance and solutions U_1 that are bounded in X_0 will grow linearly in T_0 unless the coefficients of the exponentials on the right-hand side vanish. We wish to avoid growth of the terms of our expansion with respect to T_0 because such growth ruins the asymptotic property of our expansion in the limit $\epsilon \to 0$ if T_0 is large enough. So, we demand that

$$\frac{\partial A}{\partial T_1} - 3k^2\frac{\partial A}{\partial X_1} = 0\,, \qquad \text{or} \qquad \frac{\partial A}{\partial T_1} + \omega'(k)\frac{\partial A}{\partial X_1} = 0$$

where the dispersion relation for the linearized ($\epsilon = 0$) problem is $\omega = -k^3$. This equation says that the wave envelope A propagates to the right with speed equal to $\omega'(k)$, which is called the *group velocity* associated with a wavetrain of wavenumber k. With this choice the right-hand side of the equation for U_1 becomes zero, so we may take as a solution of this equation simply

$$U_1 = 0\,.$$

With these choices, the equation for U_2 becomes

$$\begin{aligned}\frac{\partial U_2}{\partial T_0} + \frac{\partial^3 U_2}{\partial X_0^3} &= -\frac{\partial U_0}{\partial T_2} - U_0^2\frac{\partial U_0}{\partial X_0} - 3\frac{\partial^3 U_0}{\partial X_0\partial X_1^2}\\ &= -\left[\frac{\partial A}{\partial T_2} + 3ik\frac{\partial^2 A}{\partial X_1^2} + ik|A|^2 A\right]e^{i(kX_0 + k^3 T_0)} - ikA^3 e^{3i(kX_0 + k^3 T_0)} + \circledast\,.\end{aligned}$$

Now similar arguments lead us to choose the coefficient of the fundamental harmonic to be zero:

$$i\frac{\partial A}{\partial T_2} - 3k\frac{\partial^2 A}{\partial X_1^2} - k|A|^2 A = 0\,, \qquad \text{or} \qquad i\frac{\partial A}{\partial T_2} + \frac{\omega''(k)}{2}\frac{\partial^2 A}{\partial X_1^2} - k|A|^2 A = 0\,.$$

This is an NLS equation (focusing type).

The Fermi-Pasta-Ulam lattice. Virtually the same kind of analysis can be applied to the Fermi-Pasta-Ulam problem. We now want to probe the Fourier spectrum more generally than just looking in a neighborhood of $\delta = 0$. So we pick some $\delta \neq 0$ and look for solutions of the form

$$Q_n(t) = Ae^{i(\delta n - \omega t)} + A^* e^{-i(\delta n - \omega t)} + O(\epsilon)\,.$$

It turns out that if we try to just plug this in and calculate corrections for the leading-order solution, the correction terms will grow linearly in t, which would ruin the nature of our perturbation approach. So we will suppress these growing terms through the *method of multiple scales*. In addition to (but not instead of, as in the KdV derivation) the discrete spatial scale n we use the continuous variable $X = \epsilon n$. And in addition to the time variable $T_0 = t$ we also introduce extra time variables $T_1 = \epsilon t$ and $T_2 = \epsilon^2 t$. If we view $Q_n(t)$ as a function $Q = Q_n(X, T_0, T_1, T_2)$, then

$$\frac{d^2 Q_n}{dt^2} = \frac{\partial^2 Q_n}{\partial T_0^2} + 2\epsilon \frac{\partial^2 Q_n}{\partial T_0 \partial T_1} + \epsilon^2 \left[\frac{\partial^2 Q_n}{\partial T_1^2} + 2\frac{\partial^2 Q_n}{\partial T_0 \partial T_2} \right] + O(\epsilon^3)\,.$$

Also, by Taylor expansion,

$$Q_{n\pm 1}(t) = Q_{n\pm 1}(X \pm \epsilon, T_0, T_1, T_2) = Q_{n\pm 1} \pm \epsilon \frac{\partial Q_{n\pm 1}}{\partial X} + \frac{\epsilon^2}{2}\frac{\partial^2 Q_{n\pm 1}}{\partial X^2} + O(\epsilon^3)\,.$$

Finally, we introduce a perturbation expansion of Q_n itself:

$$Q_n(t) = Q_n(X, T_0, T_1, T_2) = Q_n^{(0)} + \epsilon Q_n^{(1)} + \epsilon^2 Q_n^{(2)} + O(\epsilon^3)\,.$$

Substituting these into the FPU equations for $Q_n(t)$ and separating powers of ϵ gives a collection of equations to be solved order by order:

$$L[Q^{(0)}] = 0\,,$$

$$L[Q^{(1)}] = -2m\frac{\partial^2 Q_n^{(0)}}{\partial T_0 \partial T_1} + V''(0)\left[\frac{\partial Q_{n+1}^{(0)}}{\partial X} - \frac{\partial Q_{n-1}^{(0)}}{\partial X} \right] + \frac{V'''(0)}{2}\left[(Q_{n+1}^{(0)} - Q_n^{(0)})^2 - (Q_n^{(0)} - Q_{n-1}^{(0)})^2 \right]\,,$$

and

$$L[Q^{(2)}] = \cdots$$

where $L[f]$ denotes the linear expression

$$L[f] := m\frac{\partial^2 f_n}{\partial T_0^2} - V''(0)\left[f_{n+1} - 2f_n + f_{n-1} \right]\,.$$

The leading-order equation is solved by

$$Q_n^{(0)} = Ae^{i\theta} + \circledast\,, \qquad \theta = \delta n - \omega t\,.$$

Here $A = A(X, T_1, T_2)$. This allows us to simplify the right-hand side of the equation for $Q_n^{(1)}$ which becomes

$$L[Q^{(1)}] = \left[2im\omega \frac{\partial A}{\partial T_1} + 2iV''(0)\sin(\delta)\frac{\partial A}{\partial X} \right] e^{i\theta} + iA^2 V'''(0)\left[\sin(2\delta) - 2\sin(\delta) \right] e^{2i\theta} + \circledast\,.$$

Now since $e^{i\theta}$ is a solution of the homogeneous problem, $Q_n^{(1)}$ will grow linearly in t (resonance) unless the coefficient of $e^{i\theta}$ is somehow made to be zero. We make it so by choosing the T_1 dependence of A:

$$2im\omega \frac{\partial A}{\partial T_1} + 2iV''(0)\sin(\delta)\frac{\partial A}{\partial X} = 0\,.$$

Differentiation of the dispersion relation implicitly with respect to δ reveals this equation in a simpler form:

$$\frac{\partial A}{\partial T_1} + \omega'(\delta)\frac{\partial A}{\partial X} = 0\,.$$

This says that the envelope A propagates to the right with a constant speed equal to the *group velocity* $\omega'(\delta)$.

With the dependence of A on T_1 determined, we may solve for $Q_n^{(1)}$ by the method of undetermined coefficients; seeking a solution of the form $Q_n^{(1)} = Be^{2i\theta} + \circledast$ we find

$$B = -\frac{iA^2 V'''(0)}{D(2\delta, 2\omega)}\left[\sin(2\delta) - 2\sin(\delta) \right]\,,$$

where $D(\delta,\omega) := m\omega^2 + 2V''(0)(\cos(\delta)-1)$ (the dispersion relation). Clearly it is important here that $D(2\delta,2\omega)\neq 0$ even though $D(\delta,\omega)=0$. With this choice of a particular solution for $Q_n^{(1)}$ we may write the equation for $Q_n^{(2)}$:

$$L[Q^{(2)}] = \left[2im\omega\frac{\partial A}{\partial T_2} - m\frac{\partial^2 A}{\partial T_1^2} + V''(0)\cos(\delta)\frac{\partial^2 A}{\partial X^2} - K|A|^2A\right]e^{i\theta} + \circledast + \text{second and third harmonics}\,,$$

where

$$K := 2V^{(IV)}(0)(\cos(\delta)-1)^2 + \frac{2V'''(0)^2(2\sin(\delta)-\sin(2\delta))^2}{D(2\delta,2\omega)}\,.$$

Once again, to have a bounded solution for this equation we need to kill the first harmonic terms, which gives an equation for the dependence of A on T_2:

$$2im\omega\frac{\partial A}{\partial T_2} - m\frac{\partial^2 A}{\partial T_1^2} + V''(0)\cos(\delta)\frac{\partial^2 A}{\partial X^2} - K|A|^2A = 0\,.$$

Now note that if we go into the frame of reference traveling with the group velocity:

$$\tau = T_1\,, \qquad \xi = X - \omega'(\delta)T_1\,,$$

then we have

$$\frac{\partial A}{\partial \tau} = 0\,,$$

and

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(\delta)}{2}\frac{\partial^2 A}{\partial \xi^2} + \beta|A|^2A = 0\,,$$

where

$$\beta := -\frac{m\omega^3}{4|V''(0)|^2}\left(V^{(IV)}(0) + 4V'''(0)^2\sin^2\delta\right)\,.$$

This is the nonlinear Schrödinger equation for A as a function of T_2 and X.

Derivation of NLS from the KdV equation. Make the KdV equation $\varphi_t + \varphi\varphi_x + \varphi_{xxx} = 0$ weakly nonlinear by the scaling $\varphi = \epsilon u$:

$$u_t + \epsilon u u_x + u_{xxx} = 0\,.$$

As in the analysis of the mKdV equation suppose that $u = U(X_0, X_1, T_0, T_1, T_2)$ with $X_k := \epsilon^k x$ and $T_k := \epsilon^k t$. Then by the chain rule we get

$$\frac{\partial U}{\partial T_0} + \epsilon\frac{\partial U}{\partial T_1} + \epsilon^2\frac{\partial U}{\partial T_2} + \epsilon U\frac{\partial U}{\partial X_0} + \epsilon^2 U\frac{\partial U}{\partial X_1} + \frac{\partial^3 U}{\partial X_0^3} + 3\epsilon\frac{\partial^3 U}{\partial X_0^2\partial X_1} + 3\epsilon^2\frac{\partial^3 U}{\partial X_0\partial X_1^2} + \epsilon^3\frac{\partial^3 U}{\partial X_1^3}\,.$$

Next suppose that

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + O(\epsilon^3)$$

and therefore obtain equations governing the U_k:

$$\frac{\partial U_0}{\partial T_0} + \frac{\partial^3 U_0}{\partial X_0^3} = 0\,,$$

$$\frac{\partial U_1}{\partial T_0} + \frac{\partial^3 U_1}{\partial X_0^3} = -\frac{\partial U_0}{\partial T_1} - U_0\frac{\partial U_0}{\partial X_0} - 3\frac{\partial^3 U_0}{\partial X_0^2\partial X_1}\,,$$

and

$$\frac{\partial U_2}{\partial T_0} + \frac{\partial^3 U_2}{\partial X_0^3} = -\frac{\partial U_1}{\partial T_1} - U_1\frac{\partial U_0}{\partial X_0} - U_0\frac{\partial U_1}{\partial X_0} - 3\frac{\partial^3 U_1}{\partial X_0^2\partial X_1} - \frac{\partial U_0}{\partial T_2} - U_0\frac{\partial U_0}{\partial X_1} - 3\frac{\partial^3 U_0}{\partial X_0\partial X_1^2}\,.$$

We solve the equation for U_0 as in the mKdV case:

$$U_0 = Ae^{i(kX_0 + k^3T_0)} + \circledast\,.$$

The equation for U_1 then becomes

$$\frac{\partial U_1}{\partial T_0} + \frac{\partial^3 U_1}{\partial X_0^3} = -\left[\frac{\partial A}{\partial T_1} - 3k^2\frac{\partial A}{\partial X_1}\right]e^{i(kX_0+k^3T_0)} - ikA^2e^{2i(kX_0+k^3T_0)} + \circledast\,.$$

Now, the terms proportional to the fundamental harmonics will cause resonance and so we should remove them by taking, as in the mKdV case

$$\frac{\partial A}{\partial T_1} + \omega'(k)\frac{\partial A}{\partial X_1} = 0\,,$$

where the dispersion relation is again $\omega = -k^3$. The terms that remain on the right-hand side are second harmonics. These will not cause resonance unless k is such that

$$2k^3 = -(2k)^3\,,$$

that is if $k = 0$, which we have ruled out. Therefore we do not have to worry about these terms, and we may seek a particular solution for U_1 in the form

$$U_1^{(p)} = Be^{2i(kX_0+k^3T_0)} + \circledast\,.$$

Inserting this in and solving for B we find

$$B = \frac{A^2}{6k^2}\,.$$

Now, we can add to $U_1^{(p)}$ any homogeneous solution we like, and while we might prefer to add zero and go on, it turns out that we will need to add a constant (independent of X_0 and T_0) term, which indeed satisfies the dispersion relation since $(0,0)$ is an admissable wavenumber-frequency pair. So we take

$$U_1 = \frac{A^2}{6k^2}e^{2i(kX_0+k^3T_0)} + \circledast + M$$

where the dependence of M on the slow scales is to be determined. The equation for U_2 thus becomes

$$\begin{aligned}\frac{\partial U_2}{\partial T_0} + \frac{\partial^3 U_2}{\partial X_0^3} = &-\left[-\frac{i}{6k}|A|^2A + ikMA + \frac{i}{3k}|A|^2A + \frac{\partial A}{\partial T_2} + 3ik\frac{\partial^2 A}{\partial X_1^2}\right]e^{i(kX_0+k^3T_0)} + \circledast\\ &+ \text{higher harmonics}\\ &-\frac{\partial M}{\partial T_1} - A\frac{\partial A^*}{\partial X_1} - A^*\frac{\partial A}{\partial X_1}\,.\end{aligned}$$

It should be clear now why we needed to include the M in U_1: in addition to the fundamental harmonic terms on the right-hand side we now also have "constant" terms that are also in the kernel of the operator on the left-hand side, and would cause resonant growth. Therefore to avoid such growth we should zero the coefficient of the fundamental and also the constant term. We therefore choose:

$$i\frac{\partial A}{\partial T_2} - 3k\frac{\partial^2 A}{\partial X_1^2} - \frac{1}{6k}|A|^2A - kMA = 0$$

which looks almost like a NLS equation except for the final term, and

$$\frac{\partial M}{\partial T_1} = -\frac{\partial}{\partial X_1}|A|^2\,.$$

These make up a coupled system for A and M, and to solve we also need to take into account the group-velocity propagation equation for A obtained earlier. To handle the latter, go into the group velocity frame with a change of coordinates

$$\xi = X_1 - \omega'(k)T_1\,, \qquad \tau = T_1\,.$$

Then, in the (ξ,τ) variables, the equation for group velocity propagation just reads

$$\frac{\partial A}{\partial \tau} = 0\,.$$

and at the same time the equation for M becomes

$$-\omega'(k)\frac{\partial M}{\partial \xi} + \frac{\partial M}{\partial \tau} = -\frac{\partial}{\partial \xi}|A|^2\,.$$

Combining these we can seek M as being independent of τ:

$$M = \frac{|A|^2}{\omega'(k)} = -\frac{|A|^2}{3k^2}\,.$$

Using this in the remaining equation governing the T_2 dependence of A gives a NLS equation of defocusing type:

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(k)}{2} + \frac{1}{6k}|A|^2 A = 0\,.$$

Water waves. Similar analysis applies to the water wave problem; we may derive a nonlinear Schrödinger equation by looking for a solution that is a combination of a wavetrain of wavenumber $k \neq 0$ and a "constant" term. In this problem we need to bring in the constant term at leading order.

To rewrite the water wave problem with a small parameter ϵ, assume that

$$h(x,t) = \epsilon H \qquad \text{and} \qquad \phi(x,y,t) = \epsilon\Phi\,.$$

This allows one to eliminate H from the boundary conditions at the free surface through arbitrary order in ϵ, and thus leaves a problem for Φ alone. Seeking Φ in the form

$$\Phi = \Phi_0(X_0, X_1, y, T_0, T_1, T_2) + \epsilon\Phi_1(X_0, X_1, y, T_0, T_1, T_2) + \epsilon^2\Phi_2(X_0, X_1, y, T_0, T_1, T_2) + O(\epsilon^3)$$

where the multiple scales are $X_0 = x$, $X_1 = \epsilon x$, $T_0 = t$, $T_1 = \epsilon t$, $T_2 = \epsilon^2 t$, one substitutes and solves for the Φ_n order by order. At the leading order we choose a solution of the form

$$\Phi_0 = A\cosh(k(y+1))e^{i\theta} + \circledast + M$$

where A and M are functions of X_1, T_1, and T_2 to be determined, and $\theta = kx - \omega t$ where $k \neq 0$ and ω are related by the water wave dispersion relation $\omega^2 = k\tanh(k)$. In order that Φ_1 satisfy the boundary conditions at the free surface we need to impose the solvability condition

$$\frac{\partial A}{\partial T_1} + \omega'(k)\frac{\partial A}{\partial X_1} = 0$$

which as before tells us that the wave envelope propagates to the right with the group velocity $\omega'(k)$. Then, a rather messy calculation shows that for Φ_2 to satisfy the free surface boundary condition we need to impose conditions on M and A. Eliminating M between the resulting equations gives

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(k)}{2}\frac{\partial^2 A}{\partial X_1^2} + \beta|A|^2 A = 0\,,$$

where the nonlinear coefficient β is such that $\beta\omega''(k)$ changes sign when $k \approx 1.36$. This implies a change of stability of water wave trains at a certain critical wavenumber.