Lecture 8: Local Conservation Laws from Lax Pairs

For the Toda lattice we proved that the traces of powers of the Jacobi matrix $\mathbf{L}$ are conserved quantities:

$$\frac{d}{dt}\mathrm{trace}(\mathbf{L}^p) = 0\,.$$

Since the trace is a sum over the diagonal elements, the conserved quantity $\mathrm{trace}(\mathbf{L}^p)$ can be represented as a sum of a "discrete density":

$$\mathrm{trace}(\mathbf{L}^p) = \sum_{n=0}^{N-1} \alpha_{p,n}\,,$$

where $\alpha_{p,n}$ are the diagonal elements of $\mathbf{L}^p$. In fact, one can find corresponding "discrete fluxes" $\beta_{p,n}$ such that $\beta_{p,-1} = \beta_{p,N-1} = 0$ and

$$\frac{d\alpha_{p,n}}{dt} + \beta_{p,n} - \beta_{p,n-1} = 0$$

holds for $n = 0, \ldots, N-1$. For example, when $p = 1$ we have just $\alpha_{1,n} = a_n$, and then as among the Toda lattice equations we have

$$\frac{da_0}{dt} = 2b_0^2\,, \qquad \frac{da_{N-1}}{dt} = -2b_{N-2}^2\,, \qquad \frac{da_n}{dt} = 2(b_n^2 - b_{n-1}^2)\,, \quad n = 1, \ldots, N-2\,,$$

we see that to obtain the desired local conservation law the corresponding discrete flux is $\beta_{1,n} = -2b_n^2$. In a similar way one can find discrete fluxes for all of the discrete densities $\alpha_{p,n}$.

If we multiply by $\epsilon^p/p!$ and sum all of the discrete local conservation laws over p we get a generating function for all of the local conservation laws:

$$\frac{d}{dt}\alpha_n(\epsilon) + \beta_n(\epsilon) - \beta_{n-1}(\epsilon) = 0\,,$$

where

$$\alpha_n(\epsilon) := \left(e^{\epsilon\mathbf{L}}\right)_{n,n}\,, \qquad \beta_n(\epsilon) := \sum_{p=0}^{\infty} \frac{\epsilon^p \beta_{p,n}}{p!}\,.$$

The Gardner transform revisited. We have seen that it is possible to derive an infinite number of local conservation laws for the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

with the help of the Gardner transform. Recall: one relates u to a new unknown w by

$$u = w + \epsilon w_x - \frac{\epsilon^2 w^2}{6}\,,$$

and then finds that the Gardner equation

$$w_t + \left[\frac{1}{2}w^2 - \frac{\epsilon^2}{18}w^3 + w_{xx}\right]_x = 0$$

is consistent with the KdV equation. Moreover, as w depends on an arbitrary parameter ϵ we see that w is a generating function for conserved local densities and there is a corresponding generating function for fluxes.

While this is a similar picture as we have now established for the Toda lattice problem, what is missing is a systematic derivation of the Gardner transform. In other words, how does the Gardner transform arise out of the mathematical structure that we now consider to be fundamental for KdV? At this point we consider fundamental the representation of the nonlinear KdV equation as the compatibility condition between the linear eigenvalue equation

$$L\phi = \lambda\phi\,, \qquad L\phi := -6\phi_{xx} - u\phi$$

and the linear evolution equation

$$\phi_t = B\phi := -4\phi_{xxx} - u\phi_x - \frac{1}{2}u_x\phi\,.$$

Whenever u satisfies KdV, that is, whenever the operators L and B are related through the Lax equation

$$\frac{dL}{dt} + [L, B] = 0\,,$$

the above two linear problems (the Lax pair) can be simultaneously solved for ϕ given any fixed value of $\lambda \in \mathbb{C}$. In other words, whenever u satisfies KdV, there is a function $\phi(x,t,\lambda)$ that simultaneously solves both equations of the Lax pair. The dependence on the arbitrary parameter λ is the key. This is going to be like the ϵ in the Gardner transform, as we will soon see.

Given $\phi(x,t,\lambda)$, the following equation clearly holds:

$$\frac{\partial}{\partial t} \cdot \frac{\partial}{\partial x} \log(\phi) = \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial t} \log(\phi) \,.$$

This is of the form of a local conservation law $D_t + F_x = 0$ where

$$D(x,t,\lambda) := \frac{\partial}{\partial x} \log(\phi) \,, \qquad F(x,t,\lambda) := -\frac{\partial}{\partial t} \log(\phi) \,.$$

This may look like a trivial local conservation law because the density is a derivative of something with respect to x. However, the conservation law would only be trivial if it turned out that $\log(\phi)$ could be written as a differential (with respect to x) polynomial in u. As ϕ satisfies a differential equation involving u as a coefficient, it is more likely that ϕ depends on u in a nonlocal way, and this will make the conservation law nontrivial.

Using the Lax pair, we can express the flux F as follows:

$$\begin{aligned}
F &= -\frac{\phi_t}{\phi} \\
&= 4\frac{\phi_{xxx}}{\phi} + u\frac{\phi_x}{\phi} + \frac{1}{2}u_x \\
&= -\frac{2}{3}\frac{((u+\lambda)\phi)_x}{\phi} + u\frac{\phi_x}{\phi} + \frac{1}{2}u_x \\
&= -\frac{2}{3}u_x - \frac{2}{3}(u+\lambda)\frac{\phi_x}{\phi} + u\frac{\phi_x}{\phi} + \frac{1}{2}u_x \\
&= \frac{1}{3}(u-2\lambda)\frac{\phi_x}{\phi} - \frac{1}{6}u_x \\
&= \frac{1}{3}(u-2\lambda)D - \frac{1}{6}u_x \,.
\end{aligned}$$

Therefore, the Lax pair for KdV suggests a generator for local conservation laws in the form

$$D_t + \left[\frac{1}{3}(u-2\lambda)D - \frac{1}{6}u_x\right]_x = 0 \,.$$

To realize this as a generator for local conservation laws, we need to seal the deal by somehow expanding D in a series with respect to the parameter λ and then noticing that not all of the expansion coefficients are derivatives with respect to x of differential polynomials in u (that is, are not trivial conserved local densities). To do this, we derive a differential equation satisfied by D:

$$\begin{aligned}
D_x &= (\log(\phi))_{xx} \\
&= \left(\frac{\phi_x}{\phi}\right)_x \\
&= \frac{\phi_{xx}}{\phi} - \frac{\phi_x^2}{\phi^2} \\
&= -\frac{1}{6}(u+\lambda) - \frac{\phi_x^2}{\phi^2} \\
&= -\frac{1}{6}(u+\lambda) - D^2 \,.
\end{aligned}$$

Such a first-order differential equation that is quadratic is frequently called a *Riccati equation*.

We claim that the Riccati equation for D is essentially the Gardner transform for the KdV equation. To see this, we can make a simple change of variables and a change of the parameter λ:

$$D = \frac{i\lambda^{1/2}}{\sqrt{6}}\left(1 + \frac{w}{2\lambda}\right), \qquad \epsilon := \frac{\sqrt{6}}{2i\lambda^{1/2}}.$$

Then, making these substitutions, the Riccati equation for D becomes

$$u = w + \epsilon w_x - \frac{\epsilon^2}{6} w^2 .$$

It is now clear that expansion of w for ϵ small is equivalent to expansion of D for λ large. This shows how the Gardner transform can be derived directly from the Lax pair for the KdV equation.

Conservation laws for the nonlinear Schrödinger equation. To test out this theory, we can now derive an infinite number of local conservation laws consistent with the nonlinear Schrödinger equation by following virtually the same approach as worked for KdV. The focusing and defocusing nonlinear Schrödinger equations are special cases of the zero-curvature condition that expresses the compatibility of the two first-order linear systems:

$$\frac{\partial}{\partial x}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix} i\lambda & q \\ r & -i\lambda\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix},$$

and

$$\frac{\partial}{\partial t}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix} i\lambda^2 + iqr/2 & \lambda q - iq_x/2 \\ \lambda r + ir_x/2 & -i\lambda^2 - iqr/2\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}.$$

Setting

$$D := \frac{\partial}{\partial x}\log(\phi_1), \qquad F := -\frac{\partial}{\partial t}\log(\phi_1),$$

again we automatically get the relation

$$D_t + F_x = 0,$$

and we can try to expand with respect to λ to obtain (hopefully nontrivial) local conservation laws.

First, consider the proposed flux generator, F. Using the Lax pair we can express F as follows:

$$\begin{aligned} F &= -\frac{\phi_{1t}}{\phi_1} \\ &= -i\lambda^2 - \frac{1}{2}iqr + \left(\frac{1}{2}iq_x - \lambda q\right)\frac{\phi_2}{\phi_1}. \end{aligned}$$

Now, the first row of the AKNS spectral problem, divided by ϕ_1, reads

$$D = i\lambda + q\frac{\phi_2}{\phi_1},$$

so

$$\begin{aligned} F &= -i\lambda^2 - \frac{1}{2}iqr + \left(\frac{1}{2}iq_x - \lambda q\right)\frac{D - i\lambda}{q} \\ &= \frac{\lambda q_x}{2q} - \frac{1}{2}iqr - \left(\lambda - \frac{iq_x}{2q}\right)D. \end{aligned}$$

Therefore, our proposed generator for local conservation laws reads

$$D_t + \left[\frac{\lambda q_x}{2q} - \frac{1}{2}iqr - \left(\lambda - \frac{iq_x}{2q}\right)D\right]_x = 0.$$

Now we need the nonlinear Schrödinger analogue of the Gardner transform. This arises just as in the KdV case, by obtaining a Riccati equation for D. From the Lax pair,

$$\begin{aligned} D_x &= \left(\frac{\phi_{1x}}{\phi_1}\right)_x \\ &= \left(i\lambda + q\frac{\phi_2}{\phi_1}\right)_x \\ &= q_x\frac{\phi_2}{\phi_1} + q\frac{\phi_{2x}}{\phi_1} - q\frac{\phi_2}{\phi_1}\cdot\frac{\phi_{1x}}{\phi_1} \\ &= q_x\frac{\phi_2}{\phi_1} + qr - i\lambda q\frac{\phi_2}{\phi_1} - q\frac{\phi_2}{\phi_1}\cdot\frac{\phi_{1x}}{\phi_1} \\ &= qr + (q_x - i\lambda q - qD)\frac{\phi_2}{\phi_1}\,. \end{aligned}$$

Next, we use the previously obtained expression for the quotient ϕ_2/ϕ_1 in terms of D to find

$$\begin{aligned} D_x &= qr + (q_x - i\lambda q - qD)\frac{D - i\lambda}{q} \\ &= \left(qr - i\lambda\frac{q_x}{q} - \lambda^2\right) + \frac{q_x}{q}D - D^2\,. \end{aligned}$$

This is the desired Riccati equation for D.

To obtain the local conservation laws, we need to expand Q with respect to λ using the Riccati equation. As in the KdV case, we will expand as $\lambda \to \infty$. A dominant balance argument suggests $D \approx i\lambda$ in this limit, so to begin set $D = i\lambda(1+v)$ where we expect v to be small as $\lambda \to \infty$. The Riccati equation becomes an equation for v:

$$i\lambda v_x = qr + \left(i\lambda\frac{q_x}{q} + 2\lambda^2\right)v + \lambda^2 v^2\,.$$

Now the dominant balance is to choose v to be proportional to λ^{-2} to leading order. Thus, set

$$v = -\frac{w}{2\lambda^2}\,,$$

and the equation becomes

$$qr = w + \frac{i}{2\lambda}\left(\frac{q_x}{q}w - w_x\right) - \frac{1}{4\lambda^2}w^2\,.$$

Finally, we might change the parameter λ by setting

$$\epsilon := \frac{i}{2\lambda}\,.$$

While this step is not essential, it allows us to make an analogy with the Gardner transform; indeed, now the Riccati equation has become

$$qr = w + \epsilon\left(\frac{q_x}{q}w - w_x\right) + \epsilon^2 w^2\,.$$

We might call this the "Gardner transform for the nonlinear Schrödinger equation".

Let us try this transform out, to see how it produces nontrivial local conservation laws. Substituting the expansion

$$w \sim w_0 + \epsilon w_1 + \epsilon^2 w^2 + \cdots\,, \qquad \epsilon \to 0\,,$$

and collecting powers of ϵ, we obtain

$$w_0 = qr\,,$$

and

$$w_1 = -\left(\frac{q_x}{q}w_0 - w_{0,x}\right)$$

and for $k \geq 2$ we have the recursion relation

$$w_k = -\left(\frac{q_x}{q} w_{k-1} - w_{k-1,x}\right) - \sum_{j=0}^{k-2} w_j w_{k-2-j} \,.$$

The conserved quantity associated with the nontrivial density $w_0 = qr$ is sometimes called the *norm* because when we recall the focusing and defocusing special cases:

$$\text{defocusing case: } r = q^* \,, \qquad \text{focusing case: } r = -q^* \,,$$

we see that

$$I_0[q] := \int_{-\infty}^{\infty} w_0 \, dx = \pm \int_{-\infty}^{\infty} |q(x,t)|^2 \, dx$$

which is proportional to the square of the L^2 norm. The second conserved density is

$$w_1 = (qr)_x - q_x r = q r_x \,.$$

When $r = \pm q^*$, we see that $w_1 = \pm q q_x^*$ is complex-valued. Therefore, we may further separate its real and imaginary parts:

$$\Re(w_1) = \pm\frac{1}{2}(q q_x^* + q^* q_x) = \pm\frac{1}{2}(|q|^2)_x \,,$$

which obviously gives a trivial conservation law, and

$$\Im(w_1) = \pm\frac{1}{2i}(q q_x^* - q^* q_x) \,.$$

In quantum mechanics this expression has the interpretation of momentum density, and so we refer to the corresponding conserved quantity

$$I_1[q] := \pm\frac{1}{2i}\int_{-\infty}^{\infty} (q(x,t) q_x(x,t)^* - q(x,t)^* q_x(x,t)) \, dx$$

as the *momentum*. The third conserved density is

$$w_2 = (q r_x)_x - q_x r_x - (qr)^2 = q r_{xx} - (qr)^2 \,.$$

Again this is complex when $r = \pm q^*$, so we may separate its real and imaginary parts:

$$\Re(w_2) = \pm\frac{1}{2}(q q_{xx}^* + q^* q_{xx}) - |q|^4 = \pm\frac{1}{2}(q q_x^* + q^* q_x)_x \mp |q_x|^2 - |q|^4 \,,$$

and

$$\Im(w_2) = \pm\frac{1}{2i}(q q_{xx}^* - q^* q_{xx}) = \pm\frac{1}{2i}(q q_x^* - q^* q_x)_x \,,$$

which obviously gives a trivial conservation law. Therefore at this level we learn of a new conserved quantity

$$I_2[q] := -\int_{-\infty}^{\infty} \left(\pm|q_x(x,t)|^2 + |q(x,t)|^4\right) \, dx \,.$$

This conserved quantity is the *Hamiltonian*. In other words, it turns out that the NLS equation can be written as an infinite-dimensional Hamiltonian system with respect to this Hamiltonian functional and an appropriate Poisson bracket. This procedure can be carried out to arbitrary order, and when $r = \pm q^*$, a nontrivial real local conservation law appears at each order.