Lecture 9: Exact Solutions from Lax Pairs I: Solitons

In spite of all of the structure we have described for the KdV equation

$$u_t + uu_x + u_{xxx} = 0\,,$$

we have only really seen one family of exact solutions, the solitary waves (solitons):

$$u(x,t) = 3c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right)\,.$$

We obtained this solution directly, by seeking traveling wave solutions of the form $u = f(\xi)$, $\xi = x - ct$, and therefore reducing the KdV equation to a nonlinear ordinary differential equation. We will now see how the representation of the KdV equation as the compatibility condition of a Lax pair:

$$L\phi := -6\phi_{xx} - u\phi = \lambda\phi\,, \qquad \phi_t = B\phi := -4\phi_{xxx} - u\phi_x - \frac{1}{2}u_x\phi\,,$$

allows us to re-derive this solution and obtain many other exact solutions in a systematic fashion. Analogous techniques will apply to any equation that can be written as a Lax equation or a zero-curvature condition.

A technique for deriving solutions of KdV from its Lax pair.

Lax eigenfunctions ϕ for familiar solutions of KdV.. The simplest solution of the KdV equation is $u(x,t) \equiv 0$. As this is a solution of KdV, the Lax pair equations with u replaced by 0

$$-6\phi_{xx} = \lambda\phi\,, \qquad \phi_t = -4\phi_{xxx}$$

can be simultaneously solved for ϕ. In this simple case, once we have written out the simultaneous equations for ϕ, it is obvious that they can be solved by a single function ϕ. Indeed one such solution is simply

$$\phi = e^{i(kx + 4k^3 t)}\,, \qquad \text{where} \qquad k := \sqrt{\frac{\lambda}{6}}\,.$$

Another simultaneous solution can be obtained by replacing k with $-k$.

What about when u is one of the traveling-wave soliton solutions of the KdV equation? What do the simultaneous solutions of the Lax pair admitted by the compatibility condition look like in this case? To begin with, since the soliton decays to zero for large x and t (as long as we are not in the frame moving with constant speed c), it makes sense to make a substitution

$$\phi = \psi e^{i(kx + 4k^3 t)}\,.$$

With this substitution, the Lax pair equations become

$$-6\psi_{xx} - 12ik\psi_x - u\psi = 0\,, \qquad \psi_t = -4\psi_{xxx} - 12ik\psi_{xx} + (12k^2 - u)\psi_x - \left(iku - \frac{1}{2}u_x\right)\psi\,.$$

The second equation can be simplified by using the first (twice) to convert all derivatives of ψ to first order. Thus the Lax pair equations can also be written as

$$-6\psi_{xx} - 12ik\psi_x - u\psi = 0\,, \qquad \psi_t = \left(4k^2 - \frac{1}{3}u\right)\psi_x + \left(-\frac{1}{3}iku + \frac{1}{6}u_x\right)\psi\,.$$

Now, as we are going to solve these subject to

$$u = 3c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right)\,,$$

we can go into a moving frame to simplify the equations. Moreover, since u involves hyperbolic functions, we can get simpler functions if we make a hyperbolic change of coordinates. The transformation I have in mind is to go from (x,t) to (y,τ) where

$$y = \tanh\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right)\,, \qquad \tau = t\,.$$

This maps the whole line $-\infty < x < \infty$ to the interval $-1 < y < 1$. Calculating the Jacobian with the help of the identities

$$\frac{d}{d\xi}\tanh(\xi) = \operatorname{sech}^2(\xi)\,, \qquad \tanh^2(\xi) + \operatorname{sech}^2(\xi) = 1\,,$$

we get

$$\frac{\partial}{\partial x} = \frac{\sqrt{c}}{2}(1-y^2)\frac{\partial}{\partial y}\,, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{c^{3/2}}{2}(1-y^2)\frac{\partial}{\partial y}\,.$$

Therefore,

$$\psi_x = \frac{\sqrt{c}}{2}(1-y^2)\psi_y\,,$$

$$\psi_{xx} = \frac{c}{4}(1-y^2)\frac{\partial}{\partial y}\left[(1-y^2)\psi_y\right] = \frac{c}{4}(1-y^2)\psi_{yy} - \frac{c}{2}y(1-y^2)\psi_y\,,$$

$$\psi_t = \psi_\tau - \frac{c^{3/2}}{2}(1-y^2)\psi_y\,,$$

and also,

$$u = 3c(1-y^2)\,, \qquad u_x = \frac{\sqrt{c}}{2}(1-y^2)\left[3c(1-y^2)\right]_y = -3c^{3/2}y(1-y^2)\,.$$

So, we have to simultaneously solve

$$-(1-y^2)\psi_{yy} + (3cy - 6i\sqrt{c}k)\psi_y - 3c\psi = 0\,,$$

and

$$\psi_\tau = (4k^2 + cy^2)\frac{\sqrt{c}}{2}(1-y^2)\psi_y - \left(ik + \frac{\sqrt{c}}{2}y\right)c(1-y^2)\psi\,.$$

Now, looking at the first equation, it is not too hard to see that it has a solution that is a linear function of y. Indeed, substituting $\psi = Ay + B$ where $A = A(\tau)$ and $B = B(\tau)$, we see that

$$A = -\frac{\sqrt{c}}{2ik}B\,.$$

Next, if we further suppose that B is independent of τ (say, without loss of generality, $B = ik$) then the second equation reduces to an identity.

This means that we have found a simultaneous solution of the Lax pair:

$$\phi = \left(ik - \frac{\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right)\right)e^{i(kx+4k^3t)}\,.$$

Now, let us observe an interesting property of this solution. The exponential factor $e^{i(kx+4k^3t)}$ is a part of the hyperbolic tangent if $k = \pm i\sqrt{c}/2$. Indeed,

$$\phi\bigg|_{k=\pm i\sqrt{c}/2} = \frac{\sqrt{c}}{2}\left(\mp 1 - \tanh\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right)\right)e^{\mp\sqrt{c}(x-ct)/2}\,.$$

These two functions of x and t are in fact proportional to each other:

$$\frac{\phi(k = i\sqrt{c}/2)}{\phi(k = -i\sqrt{c}/2)} = -e^{-\sqrt{c}x_0}\,.$$

This is a key feature that we can try to emulate in generating more complicated solutions of KdV from the Lax pair.

Generalization. Perhaps for some other solution of KdV, there exist simultaneous solutions ϕ of the Lax pair having the form

$$(1)\quad \phi(x,t,k) = ((ik)^N + A_{N-1}(x,t)(ik)^{N-1} + A_{N-2}(x,t)(ik)^{N-2} + \cdots + A_1(x,t)(ik) + A_0(x,t))e^{i(kx+4k^3t)}\,,$$

for some $N = 0, 1, 2, 3, \ldots$. We will suppose that this function satisfies the following relations:

$$(2)\qquad \phi(x,t,i\kappa_n) = (-1)^n c_n \phi(x,t,-i\kappa_n)\,, \qquad n = 1,\ldots,N\,,$$

where $0 < \kappa_1 < \kappa_2 < \cdots < \kappa_{N-1} < \kappa_N$ and where $c_n > 0$ for all n. Indeed, this was the case for $N = 0$ (yielding the trivial solution $u = 0$ of KdV) and $N = 1$ (yielding the soliton solution of KdV).

The first important observation is that the N homogeneous conditions (2) actually determine $\phi(x,t,k)$ uniquely for all $(x,t) \in \mathbb{R}^2$. This is a consequence of the following more general result.

Proposition 1. *For each $(x,t) \in \mathbb{R}^2$, the set Λ of functions $f(x,t,k)$ of a complex variable k having the form*

$$f(x,t,k) = \left((ik)^N F_N + (ik)^{N-1}F_{N-1} + \cdots + (ik)F_1 + F_0\right)e^{i(kx+4k^3t)}$$

is a vector space of dimension $N+1$ over $\mathbb{C}$. Given a set of numbers $0 < \kappa_1 < \kappa_2 < \cdots < \kappa_{N-1} < \kappa_N$ and $c_n > 0$, for $n = 1,\ldots,N$, the set of elements of Λ that satisfy the homogeneous conditions (2) form a linear subspace Λ_0. Moreover

$$\dim(\Lambda_0) = 1\,,$$

for all $(x,t) \in \mathbb{R}^2$. In particular, this means that if $f \in \Lambda_0$ is normalized by assuming $F_N = 1$, then it is uniquely determined, and that if $f \in \Lambda_0$ satisfies $F_N = 0$, then $f(x,t,k) \equiv 0$.

Proof. The fact that Λ is a complex vector space of dimension $N+1$ is obvious (the coefficients F_n could be taken as coordinates). Now as the conditions (2) are linear and homogeneous in the coefficients F_n, it is clear that they carve out a linear subspace of Λ. The issue is the independence of these N conditions; if they are all independent then each condition reduces the dimension by one, and we will have $\dim(\Lambda_0) = 1$. The fact that the conditions are indeed all independent for all $(x,t) \in \mathbb{R}^2$ amounts to a determinant being nonzero for all $(x,t) \in \mathbb{R}^2$. We will prove that this determinant is nonzero later, by directly showing that the conditions (1) and (2) uniquely determine $\phi(x,t,k)$ for all $(x,t) \in \mathbb{R}^2$ and all $k \in \mathbb{C}$. ☐

So, for now we understand that the function $\phi(x,t,k)$ defined by (1) and (2) is well-defined, and therefore so are the coefficients $A_0, A_1, \ldots, A_{N-1}$, which are functions of $(x,t) \in \mathbb{R}^2$. We will now show that this function satisfies some linear differential equations.

Proposition 2. *The function $\phi(x,t,k)$ satisfies*

$$L\phi := -6\phi_{xx} - u\phi = \lambda\phi\,,$$

where $\lambda = 6k^2$, and where the potential function $u(x,t)$ is given by

$$u(x,t) = -12\frac{\partial A_{N-1}}{\partial x}\,.$$

Proof. Consider the function $w(x,t,k)$ defined by

$$w := L\phi - \lambda\phi\,,$$

where for the moment we think of the potential $u(x,t)$ as being arbitrary. The function $w(x,t,k)$ is, for each $(x,t) \in \mathbb{R}^2$, an element of the space Λ_0. Indeed, by direct calculation, we have

$$\begin{aligned} w =& 6k^2\left((ik)^N + (ik)^{N-1}A_{N-1} + \cdots + A_0\right)e^{i(kx+4k^3t)} \\ &- 12ik\left((ik)^{N-1}A_{N-1,x} + (ik)^{N-2}A_{N-2,x} + \cdots + A_{0,x}\right)e^{i(kx+4k^3t)} \\ &- 6\left((ik)^{N-1}A_{N-1,xx} + \cdots + A_{0,xx}\right)e^{i(kx+4k^3t)} \\ &- u\left((ik)^N + (ik)^{N-1}A_{N-1} + \cdots + A_0\right)e^{i(kx+4k^3t)} \\ &- 6k^2\left((ik)^N + (ik)^{N-1}A_{N-1} + \cdots + A_0\right)e^{i(kx+4k^3t)}\,, \\ =& \left((ik)^N W_N + (ik)^{N-1}W_{N-1} + \cdots + W_0\right)e^{i(kx+4k^3t)}\,, \end{aligned}$$

where

$$W_N := -12A_{N-1,x} - u\,, \qquad W_{N-1} := -12A_{N-2,x} - 6A_{N-1,xx} - uA_{N-1}\,,$$

and so on. This shows immediately that w is an element of the vector space Λ. To see that it belongs to the subspace Λ_0 it therefore remains to check whether $w(x,t,k)$ satisfies the homogeneous relations (2). But this follows from the fact that $\phi(x,t,k)$ satisfies these relations for all $(x,t) \in \mathbb{R}^2$, and that the homogeneous conditions (2) that isolate the subspace Λ_0 of Λ are all independent of x and t.

Now we note that if the potential u is not arbitrary, but is taken to be related to the coefficient A_{N-1} by $u = -12A_{N-1,x}$, then $W_N = 0$ for all x and t, and consequently (because Λ_0 is one-dimensional, and so all elements are proportional to $\phi(x,t,k)$ by a factor that is a function of x and t) $w(x,t,k) \equiv 0$. This completes the proof. □

We note in passing that the proof of this proposition also tells us that $W_{N-1} = 0$, or in other words,

$$0 = -12A_{N-2,x} - 6A_{N-1,xx} - uA_{N-1} = -12A_{N-2,x} - 6A_{N-1,xx} + 12A_{N-1}A_{N-1,x}\,. \tag{3}$$

Also, we have the following.

Proposition 3. *The function $\phi(x,t,k)$ satisfies*

$$\phi_t = B\phi := -4\phi_{xxx} - u\phi_x - \frac{1}{2}u_x\phi\,,$$

where again the potential function $u(x,t)$ is given by

$$u(x,t) = -12\frac{\partial A_{N-1}}{\partial x}\,.$$

Proof. We now let $z(x,t,k)$ be defined as

$$z := \phi_t - B\phi\,.$$

As in the proof of Proposition 2, $z(x,t,k)$ satisfies the same homogeneous relations (2) as does $\phi(x,t,k)$. Now we write out z:

$$\begin{aligned}
z =\,& 4ik^3\left((ik)^N + (ik)^{N-1}A_{N-1} + \cdots + A_0\right)e^{i(kx+4k^3t)}\\
&+ \left((ik)^{N-1}A_{N-1,t} + \cdots + A_{0,t}\right)e^{i(kx+4k^3t)}\\
&- 4ik^3\left((ik)^N + (ik)^{N-1}A_{N-1} + \cdots + A_0\right)e^{i(kx+4k^3t)}\\
&- 12k^2\left((ik)^{N-1}A_{N-1,x} + (ik)^{N-2}A_{N-2,x} + \cdots + A_{0,x}\right)e^{i(kx+4k^3t)}\\
&+ 12ik\left((ik)^{N-1}A_{N-1,xx} + \cdots + A_{0,xx}\right)e^{i(kx+4k^3t)}\\
&+ 4\left((ik)^{N-1}A_{N-1,xxx} + \cdots + A_{0,xxx}\right)e^{i(kx+4k^3t)}\\
&+ iku\left((ik)^N + (ik)^{N-1}A_{N-1} + \cdots + A_0\right)e^{i(kx+4k^3t)}\\
&+ u\left((ik)^{N-1}A_{N-1,x} + \cdots + A_{0,x}\right)e^{i(kx+4k^3t)}\\
&+ \frac{1}{2}u_x\left((ik)^N + (ik)^{N-1}A_{N-1} + \cdots + A_0\right)e^{i(kx+4k^3t)}\\
=\,& \left((ik)^{N+1}Z_{N+1} + (ik)^N Z_N + (ik)^{N-1}Z_{N-1} + \cdots + Z_0\right)e^{i(kx+4k^3t)}\,,
\end{aligned}$$

where

$$Z_{N+1} := 12A_{N-1,x} + u\,, \qquad Z_N := 12A_{N-2,x} + 12A_{N-1,xx} + uA_{N-1} + \frac{1}{2}u_x\,,$$

and so on. Now if we specify u to be $u = -12A_{N-1,x}$, then obviously $Z_{N+1} = 0$. At this point we have proved that z is, for each $(x,t) \in \mathbb{R}^2$, an element of the one-dimensional linear space Λ_0. If we can establish that $Z_N = 0$ as well, then we will have proved that $z(x,t,k) \equiv 0$. But the fact that $Z_N = 0$ follows from (3) along with $u = -12A_{N-1,x}$. This completes the proof. □

Of course the two linear equations satisfied by $\phi(x,t,k)$ are none other than the Lax pair for KdV with $u = -12A_{N-1,x}$. As we have produced a simultaneous solution $\phi(x,t,k)$ of the KdV Lax pair for this u, it is necessary that $u = -12A_{N-1,x}$ satisfy the KdV equation. Therefore we have proved the following.

Corollary 1. *Given positive numbers $0 < \kappa_1 < \kappa_2 < \cdots < \kappa_N$ and positive numbers $c_1, \ldots, c_N$, let the coefficient $A_{N-1}(x,t)$ be constructed by imposing the homogeneous conditions* (2) *on the form* (1)*. Then the function*

$$u(x,t) := -12\frac{\partial A_{N-1}}{\partial x}$$

is a solution of the KdV equation $u_t + uu_x + u_{xxx} = 0$.

Note: in principle, the conditions (2) may be generalized. The main properties we needed were that for all $(x,t) \in \mathbb{R}^2$ the conditions were independent, and that they were linear, homogeneous, and independent of $(x,t) \in \mathbb{R}^2$. The precise structure of these conditions, namely evaluation at opposite points on the imaginary axis, and the reality of $\pm c_n$ and the alternating pattern of signs, are important only in making sure that $u(x,t)$ is real-valued for real x and t, and also in making sure that $u(x,t)$ does not contain any singularities for real x and t. As such, constraints that ensure that the homogeneous relations used to isolate the subspace Λ_0 within the larger space Λ lead to a nonsingular, real-valued solution of KdV are called *reality conditions*. As an exercise, consider $N = 1$ and replace the condition $\phi(x,t,i\kappa_1) = -c_1\phi(x,t,-i\kappa_1)$ with $c_1 > 0$ by $\phi(x,t,i\kappa_1) = c_1\phi(x,t,-i\kappa_1)$; solve for $A_0(x,t)$ and show that $u = -12A_0(x,t)$ is real-valued and satisfies KdV, but has a singularity that moves at a constant speed.

The Kay-Moses formula. To make this procedure effective we need to go into more details about how to solve for A_{N-1} given $0 < \kappa_1 < \kappa_2 < \cdots < \kappa_N$ and $c_1, \ldots, c_N$ all positive. In doing so we will prove that the conditions (2) are indeed independent for each $(x,t) \in \mathbb{R}^2$ and that the function $A_{N-1}(x,t)$ is real-valued for all $(x,t) \in \mathbb{R}^2$. The formula we will arrive at was obtained by Kay and Moses in 1956; they were studying the problem of determining "reflectionless" potentials for the Schrödinger operator, and these turn out to be exactly the solutions of KdV that we have just established (although Kay and Moses were unaware of the connection to KdV).

Let $Q(k)$ denote the polynomial

$$Q(k) := (ik)^N + (ik)^{N-1}A_{N-1} + (ik)^{N-2}A_{N-2} + \cdots + ikA_1 + A_0\,.$$

Then, the conditions (2) can be written in the form

$$Q(i\kappa_n) = (-1)^n E_n^2 Q(-i\kappa_n)\,, \qquad n = 1, \ldots, N\,, \tag{4}$$

where

$$E_n := e^{\kappa_n x - 4\kappa_n^3 t + \alpha_n}\,, \qquad e^{\alpha_n} := \sqrt{c_n} > 0\,.$$

We can obtain an alternate representation of $Q(k)$ by writing it in terms of its values $\{Q(i\kappa_n)\}$ at the points $\{i\kappa_n\}$, rather than writing it in terms of its coefficients $\{A_n\}$. Writing

$$w_n := Q(i\kappa_n)\,, \qquad n = 1, \ldots, N\,,$$

the Lagrange interpolation formula gives

$$Q(k) = (ik)^N + \sum_{n=1}^{N}(w_n - (-\kappa_n)^N)\prod_{\substack{m=1\\m\neq n}}^{N}\frac{k - i\kappa_m}{i\kappa_n - i\kappa_m}\,.$$

Note further that

$$(ik)^N - \sum_{n=1}^{N}(-\kappa_n)^N\prod_{\substack{m=1\\m\neq n}}^{N}\frac{k - i\kappa_m}{i\kappa_n - i\kappa_m}\,.$$

is a polynomial of degree N with leading coefficient $(ik)^N$ that vanishes for $k = i\kappa_1, \ldots, i\kappa_N$. Therefore we have an identity

$$(ik)^N - \sum_{n=1}^{N}(-\kappa_n)^N\prod_{\substack{m=1\\m\neq n}}^{N}\frac{k - i\kappa_m}{i\kappa_n - i\kappa_m} = i^N\prod_{n=1}^{N}(k - i\kappa_n)\,.$$

So from Lagrange interpolation and this observation, we can write $Q(k)$ in terms of its values w_n at $k = i\kappa_n$ as

$$Q(k) = i^N \prod_{n=1}^{N}(k - i\kappa_n) + \sum_{n=1}^{N} w_n \prod_{\substack{m=1\\m\neq n}}^{N} \frac{k - i\kappa_m}{i\kappa_n - i\kappa_m}\,.$$

Now we substitute this into (4) to obtain a system of linear equations for $\{w_n\}$:

$$w_n = (-1)^n E_n^2 \prod_{m=1}^{N}(\kappa_n + \kappa_m) + (-1)^{n+N-1} E_n^2 \sum_{m=1}^{N} (-1)^{N-m} w_m \prod_{\substack{\ell=1\\ \ell\neq m}}^{N} \frac{\kappa_n + \kappa_\ell}{|\kappa_m - \kappa_\ell|}\,, \qquad n = 1, \ldots, N\,.$$

A simple transformation symmetrizes this system:

$$w_n = (-1)^n E_n \left[\prod_{m=1}^{N}(\kappa_n + \kappa_m) \cdot \prod_{\substack{m=1\\m\neq n}}^{N} |\kappa_n - \kappa_m|\right]^{1/2} a_n\,, \qquad n = 1, \ldots, N\,,$$

yielding

$$a_n + \sum_{m=1}^{N} \frac{F_n F_m}{\kappa_n + \kappa_m} a_m = F_n\,,$$

where

$$F_n := \frac{\displaystyle\prod_{m=1}^{N}(\kappa_n + \kappa_m)}{\displaystyle\prod_{\substack{m=1\\m\neq n}}^{N} |\kappa_n - \kappa_m|} E_n = e^{\kappa_n x - 4\kappa_n^3 t + \beta_n}\,,$$

for some $\beta_n \in \mathbb{R}$.

Now what we need is not really the $\{a_n\}$ or even the $\{w_n\}$, but rather the coefficient A_{N-1} of $(ik)^{N-1}$ in $Q(k)$. From the Lagrange interpolation formula,

$$Q(k) = i^N k^N - i^{N+1} k^{N-1} \sum_{n=1}^{N} \kappa_n + k^{N-1} \sum_{n=1}^{N} w_n \prod_{\substack{m=1\\m\neq n}}^{N} \frac{1}{i\kappa_n - i\kappa_m} + O(k^{N-2})\,,$$

as $k \to \infty$, so

$$\begin{aligned} A_{N-1} &= \sum_{n=1}^{N} \kappa_n + (-1)^{N-1} \sum_{n=1}^{N} \frac{w_n}{\displaystyle\prod_{\substack{m=1\\m\neq n}}^{N} (\kappa_n - \kappa_m)} \\ &= \sum_{n=1}^{N} \kappa_n - \sum_{n=1}^{N} F_n a_n\,. \end{aligned}$$

We now consider solving for the $\{a_n\}$ using Cramer's rule, and then writing down the sum

$$S := \sum_{n=1}^{N} F_n a_n\,,$$

that appears as the nonconstant term in the formula for A_{N-1}. By Cramer's rule,

$$F_n a_n = F_n \frac{\det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k} \quad \text{with column } n \text{ replaced by} \quad (F_1, F_2, \ldots, F_N)^T\right)}{\det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right)}\,.$$

Now the determinant of a matrix is linear in any one column of the matrix, so we may pass the F_n through to the nth column of the matrix whose determinant is in the numerator:

$$F_n a_n = \frac{\det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k} \quad \text{with column } n \text{ replaced by} \quad (F_n F_1, F_n F_2, \ldots, F_n F_N)^T\right)}{\det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right)} .$$

Now note that the nth column of the coefficient matrix is

$$\mathbf{e}_n + \begin{bmatrix} \frac{F_1 F_n}{\kappa_1 + \kappa_n} \\ \frac{F_2 F_n}{\kappa_2 + \kappa_n} \\ \vdots \\ \frac{F_N F_n}{\kappa_N + \kappa_n} \end{bmatrix}$$

and the x-derivative of this is just $(F_1 F_n, F_2 F_n, \ldots, F_N F_n)^T$ by definition of F_j. Therefore,

$$F_n a_n = \frac{\det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k} \quad \text{with column } n \text{ replaced by its } x\text{-derivative}\right)}{\det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right)} .$$

Now we have another identity for determinants, which follows by linearity in each column separately, namely

$$\Big[\det(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(N)})\Big]_x = \det(\mathbf{v}^{(1)}_x, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(N)}) + \det(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}_x, \ldots, \mathbf{v}^{(N)}) + \cdots + \det(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(N)}_x) .$$

From this identity it follows that

$$S = \sum_{n=1}^{N} F_n a_n = \frac{\det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right)_x}{\det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right)} = \frac{\partial}{\partial x} \log\det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right) .$$

Therefore,

$$A_{N-1} = \sum_{n=1}^{N} \kappa_n - \frac{\partial}{\partial x} \log\det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right) ,$$

and so a solution of KdV is given by

$$u(x,t) = -12 A_{N-1,x} = 12 \frac{\partial^2}{\partial x^2} \log(\tau) , \qquad \text{where} \qquad \tau := \det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right) .$$

In using this formula, it is more natural to think of the data $0 < \kappa_1 < \kappa_2 < \cdots < \kappa_N$ and arbitrary real constants $\beta_1, \ldots, \beta_N$ to be given (the $\{\beta_n\}$ rather than the $\{c_n\}$).

One question we should definitely ask at this point is whether τ can be equal to zero for any $(x,t) \in \mathbb{R}^2$. The determinant τ being nonzero is equivalent to the assertion in Proposition 1 that the homogeneous conditions (2) are independent. Now, τ can be viewed as the characteristic polynomial $P(\lambda)$ of the matrix with entries $F_j F_k/(\kappa_j + \kappa_k)$ evaluated at $\lambda = -1$. We can calculate the coefficients of the various powers of λ in

$$P(\lambda) := \det\left(-\lambda\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right)$$

by thinking of the problem as a perturbation problem for large λ. In other words, as $\lambda \to \infty$, we have

$$P(\lambda) = (-\lambda)^N + (-\lambda)^{N-1} P_{N-1} + (-\lambda)^{N-2} P_{N-2} + \cdots + (-\lambda) P_1 + P_0 ,$$

and we can compute the $\{P_k\}$ perturbatively. Once we have the coefficients, then

$$\tau = \det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right) = P(-1) = 1 + P_{N-1} + P_{N-2} + \cdots + P_1 + P_0 .$$

It is not too hard to see that

$$P_k = \sum_{\text{subsets } S \text{ of } \{1,\ldots,N\} \text{ of size } N-k} \det\left(\left.\frac{F_\alpha F_\beta}{\kappa_\alpha + \kappa_\beta}\right|_{\alpha,\beta \in S}\right) ,$$

and therefore

$$\tau = \det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right) = 1 + \sum_{S\subset\{1,\dots,N\}} \det\left(\left.\frac{F_\alpha F_\beta}{\kappa_\alpha+\kappa_\beta}\right|_{\alpha,\beta\in S}\right).$$

We call this the *principal minors expansion of* τ. We claim that all of the terms of this sum are positive, which ensures that the determinant is nonzero. First of all, since the F_α and F_β come in as diagonal factors on the left and right:

$$\left(\left.\frac{F_\alpha F_\beta}{\kappa_\alpha+\kappa_\beta}\right|_{\alpha,\beta\in S}\right) = \mathrm{diag}(F_{s_1}, F_{s_2}, \dots, F_{s_{|S|}})\left(\left.\frac{1}{\kappa_\alpha+\kappa_\beta}\right|_{\alpha,\beta\in S}\right)\mathrm{diag}(F_{s_1}, F_{s_2}, \dots, F_{s_{|S|}}),$$

so

$$\det\left(\left.\frac{F_\alpha F_\beta}{\kappa_\alpha+\kappa_\beta}\right|_{\alpha,\beta\in S}\right) = F_{s_1}^2 F_{s_2}^2 \cdots F_{s_{|S|}}^2 \det\left(\left.\frac{1}{\kappa_\alpha+\kappa_\beta}\right|_{\alpha,\beta\in S}\right).$$

Furthermore, the determinant that remains can be calculated explicitly:

$$\det\left(\left.\frac{1}{\kappa_\alpha+\kappa_\beta}\right|_{\alpha,\beta\in S}\right) = \frac{\displaystyle\prod_{\substack{\alpha,\beta\in S\\ \alpha\neq\beta}} |\kappa_\alpha - \kappa_\beta|}{\displaystyle\prod_{\alpha,\beta\in S}(\kappa_\alpha+\kappa_\beta)}.$$

Therefore it is indeed true that the terms in the sum are all positive so $\tau \geq 1$, which actually completes the proof of Proposition 1.

Multisoliton solutions. What is the character of the solutions of KdV given by the Kay-Moses determinantal formula? We already know that if $N = 1$ this procedure reproduces the soliton solution of KdV. We will show now that for $N > 1$ the Kay-Moses formula gives solutions describing the nonlinear interaction of N solitons.

Just to get started, let's evaluate the Kay-Moses formula for $N = 1$ and see how the soliton arises from it. When $N = 1$, we have explicitly

$$\tau = 1 + \frac{F_1^2}{2\kappa_1}.$$

This is of the more general form

$$\tau = 1 + g_n^2 F_n^2,$$

where n is arbitrary and g_n is a real constant. From this formula, we get

$$12\frac{\partial^2}{\partial x^2}\log(\tau) = 12\frac{\partial}{\partial x}\frac{2\kappa_n g_n^2 F_n^2}{1+g_n^2F_n^2} = 12\frac{\partial}{\partial x}\left[\kappa_n + \kappa_n \tanh\left(\kappa_n x - 4\kappa_n^3 t + \beta_n + \gamma_n\right)\right],$$

where $e^{\gamma_n} = |g_n|$. Therefore in this case,

$$12\frac{\partial^2}{\partial x^2}\log(\tau) = 12\kappa_n^2\mathrm{sech}^2\left(\kappa_n x - 4\kappa_n^3 t + \beta_n + \gamma_n\right).$$

Now, let's consider $N > 1$, and place ourselves in a frame of reference moving with speed $c = 4\kappa_n^2$ for some $n \in 1, 2, \dots, N$. So, we take

$$\xi := x - 4\kappa_n^2 t$$

to be held fixed and we will let t vary between $\pm\infty$ and see what happens to τ, and by extension, to the corresponding solution u of the KdV equation. Note that

$$\kappa_j x - 4\kappa_j^3 t = \kappa_j \xi + 4\kappa_j(\kappa_n^2 - \kappa_j^2)t,$$

so that as $t \to -\infty$ with ξ fixed, we have

$$F_j \to \infty, \quad \text{for } j < n \qquad \text{and} \qquad F_j \to 0, \quad \text{for } j > n$$

while as $t \to +\infty$ with ξ fixed, we have

$$F_j \to 0, \quad \text{for } j < n \qquad \text{and} \qquad F_j \to \infty, \quad \text{for } j > n.$$

Note that as $F_n = \kappa_n \xi + \beta_n$ is independent of t it remains fixed in both limits. Also, these limits are valid in the C^k sense, in that they remain true if F_j is replaced by any number of derivatives with respect to x.

Let us write the principal minors expansion of τ in a form that will be advantageous for considering the limit $t \to -\infty$:

$$\tau = F_1^2 F_2^2 \cdots F_{n-1}^2 \left\{ \frac{1}{F_1^2 F_2^2 \cdots F_{n-1}^2} + \sum_{S \subset \{1,\ldots,N\}} \frac{F_{s_1}^2 F_{s_2}^2 \cdots F_{s_{|S|}}^2}{F_1^2 F_2^2 \cdots F_{n-1}^2} \frac{\displaystyle\prod_{\substack{\alpha,\beta \in S \\ \alpha \neq \beta}} |\kappa_\alpha - \kappa_\beta|}{\displaystyle\prod_{\alpha,\beta \in S} (\kappa_\alpha + \kappa_\beta)} \right\} .$$

Now, as $t \to -\infty$, all of the terms in the braces will be going to zero along with their derivatives *with the exception of those corresponding to the subsets* $S = \{1, 2, 3, \ldots, n-1\}$ *and* $S = \{1, 2, 3, \ldots, n\}$. Therefore we may write

$$\tau = F_1^2 F_2^2 \cdots F_{n-1}^2 \left\{ \frac{\displaystyle\prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{n-1} |\kappa_\alpha - \kappa_\beta|}{\displaystyle\prod_{\alpha,\beta=1}^{n-1} (\kappa_\alpha + \kappa_\beta)} + F_n \frac{\displaystyle\prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{n} |\kappa_\alpha - \kappa_\beta|}{\displaystyle\prod_{\alpha,\beta=1}^{n} (\kappa_\alpha + \kappa_\beta)} + o(1) \right\} , \qquad \text{as } t \to -\infty .$$

Alternately, we may write this in the form

$$\tau = F_1^2 F_2^2 \cdots F_{n-1}^2 \frac{\displaystyle\prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{n-1} |\kappa_\alpha - \kappa_\beta|}{\displaystyle\prod_{\alpha,\beta=1}^{n-1} (\kappa_\alpha + \kappa_\beta)} \left\{ 1 + \left[\frac{1}{2\kappa_n} \prod_{k=1}^{n-1} \left(\frac{\kappa_n - \kappa_k}{\kappa_n + \kappa_k} \right)^2 \right] F_n^2 + o(1) \right\} , \qquad \text{as } t \to -\infty .$$

Now, noting that $\log(F_k)$ is a linear expression in ξ and that $\partial/\partial x = \kappa_n \partial/\partial \xi$, we get

$$u(x,t) = 12 \frac{\partial^2}{\partial x^2} \log(\tau) = 12 \kappa_n^2 \operatorname{sech}^2 \left(\kappa_n x - 4\kappa_n^3 t + \beta_n + \gamma_n^- \right) + o(1) ,$$

as $t \to -\infty$ with $x - 4\kappa_n^2 t = \xi$ held fixed, where

$$\gamma_n^- := \log \left(\frac{1}{\sqrt{2\kappa_n}} \right) + \sum_{k=1}^{n-1} \log \left(\frac{|\kappa_n - \kappa_k|}{\kappa_n + \kappa_k} \right) .$$

In a similar fashion we can examine the asymptotic behavior of the solution as $t \to +\infty$. Here a form of τ that is advantageous for calculations is

$$\tau = F_{n+1}^2 F_{n+2}^2 \cdots F_N^2 \left\{ \frac{1}{F_{n+1}^2 F_{n+2}^2 \cdots F_N^2} + \sum_{S \subset \{1,\ldots,N\}} \frac{F_{s_1}^2 F_{s_2}^2 \cdots F_{s_{|S|}}^2}{F_{n+1}^2 F_{n+2}^2 \cdots F_N^2} \frac{\displaystyle\prod_{\substack{\alpha,\beta \in S \\ \alpha \neq \beta}} |\kappa_\alpha - \kappa_\beta|}{\displaystyle\prod_{\alpha,\beta \in S} (\kappa_\alpha + \kappa_\beta)} \right\} .$$

Now the only terms in the braces that will not be small as $t \to +\infty$ are those corresponding to $S = \{n, n+1, \dots, N\}$ and $S = \{n+1, n+2, \dots, N\}$, so we have

$$\tau = F_{n+1}^2 F_{n+2}^2 \cdots F_N^2 \left\{ \frac{\displaystyle\prod_{\substack{\alpha,\beta=n+1 \\ \alpha\neq\beta}}^N |\kappa_\alpha - \kappa_\beta|}{\displaystyle\prod_{\alpha,\beta=n+1}^N (\kappa_\alpha - \kappa_\beta)} + F_n \frac{\displaystyle\prod_{\substack{\alpha,\beta=n \\ \alpha\neq\beta}}^N |\kappa_\alpha - \kappa_\beta|}{\displaystyle\prod_{\alpha,\beta=n}^N (\kappa_\alpha - \kappa_\beta)} + o(1) \right\}, \qquad \text{as } t \to +\infty\,,$$

or

$$\tau = F_{n+1}^2 F_{n+2}^2 \cdots F_N^2 \frac{\displaystyle\prod_{\substack{\alpha,\beta=n+1 \\ \alpha\neq\beta}}^N |\kappa_\alpha - \kappa_\beta|}{\displaystyle\prod_{\alpha,\beta=n+1}^N (\kappa_\alpha - \kappa_\beta)} \left\{ 1 + \left[\frac{1}{2\kappa_n} \prod_{k=n+1}^N \left(\frac{\kappa_n - \kappa_k}{\kappa_n + \kappa_k} \right)^2 \right] F_n + o(1) \right\}, \qquad \text{as } t \to +\infty\,.$$

From this it follows that

$$u(x,t) = 12\kappa_n^2 \mathrm{sech}^2 \left(\kappa_n x - 4\kappa_n^3 t + \beta_n + \gamma_n^+ \right) + o(1)\,,$$

as $t \to +\infty$ with $\xi = x - 4\kappa_n^2 t$ held fixed, where

$$\gamma_n^+ := \log\left(\frac{1}{\sqrt{2\kappa_n}} \right) + \sum_{k=n+1}^N \log\left(\frac{|\kappa_n - \kappa_k|}{\kappa_n + \kappa_k} \right)\,.$$

Therefore, in the frame of reference traveling with the velocity $c = 4\kappa_n^2$, we see an isolated soliton of the KdV equation in both limits $t \to \pm\infty$. By looking at the traveling frames of reference corresponding to all $n = 1, \dots, N$, we see that as $t \to \pm\infty$ the solution $u(x,t)$ decouples into a linear sum of elementary solitons:

$$u(x,t) = \sum_{n=1}^N 12\kappa_n^2 \mathrm{sech}^2 \left(\kappa_n x - 4\kappa_n^3 t + \beta_n + \gamma_n^\pm \right)\,, \qquad \text{as } t \to \pm\infty\,.$$

Based on this observation we refer to such solutions $u(x,t)$ as *multisoliton* or *N-soliton* solutions. Note that the decomposition into solitons is only valid as $t \to \pm\infty$; for finite t the solitons are indistinguishable from one another and undergo a complicated nonlinear interaction.

The only difference between the asymptotic wave forms at $t = \pm\infty$ enters through the constants $\gamma_n^\pm$. As a result of its nonlinear interactions with the $N-1$ other solitons that occur for finite t, the soliton is boosted forward in the spatial variable x by an amount

$$\Delta x_n := \frac{\gamma_n^- - \gamma_n^+}{\kappa_n} = \frac{1}{\kappa_n} \sum_{k=1}^{n-1} \log\left(\frac{|\kappa_n - \kappa_k|}{\kappa_n + \kappa_k} \right) - \frac{1}{\kappa_n} \sum_{k=n+1}^N \log\left(\frac{|\kappa_n - \kappa_k|}{\kappa_n + \kappa_k} \right)\,.$$

This is a formula for the asymptotic *phase shift* experienced by the nth soliton as a result of its interaction with the remaining solitons. Note that this formula only depends on the speed/amplitude parameters $\{\kappa_k\}$ and not on any phase information encorporated in the parameters $\{\beta_k\}$. Therefore the same asymptotic phase shift results whether the solitons interact more or less pairwise or whether they interact all together.