

**Periodic potentials for the Schrödinger operator.** Another class of solutions for KdV arose in connection with thinking of what would happen under the KdV flow to an initial condition  $u(x)$  that is a periodic function of  $x$ . From the point of view of inverse-scattering theory, we should think first about the problem of computing spectral data for the periodic Schrödinger operator:

$$-6\phi_{xx} + u\phi = \lambda\phi, \quad u(x+L) = u(x).$$

The spectrum  $S$  consists of the values of  $\lambda \in \mathbb{C}$  for which there exist solutions  $\phi$  that are bounded for all  $x$ . An example that is well-studied is *Mathieu's equation*

$$-6\phi_{xx} + \delta \cos(x)\phi = \lambda\phi,$$

where  $\delta$  is a real parameter. In this case, it is known that (see below) when  $\delta = 0$  the spectrum consists of the positive half-line:

$$S = [0, +\infty), \quad \text{for the Mathieu equation with } \delta = 0,$$

while for all  $\delta \neq 0$  an infinite number of gaps open up in the spectrum. The gaps become exponentially small in length as  $\lambda \rightarrow +\infty$ , but they occur infinitely often in this limit. For  $\delta \neq 0$ , the spectrum consists of an infinite union of closed intervals

$$S = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup [\lambda_4, \lambda_5] \cup \dots \quad \text{for } \delta \neq 0.$$

This sort of situation is somewhat typical for periodic potentials. However, it was also known that for certain very special periodic potentials it could turn out that there were only a finite number of gaps in the spectrum. For example, it was known that if  $u(x)$  was a certain sort of elliptic function (we will derive this below), then the spectrum has only one gap in it, and so has the form

$$S = [\lambda_0, \lambda_1] \cup [\lambda_2, +\infty), \quad \text{for a family of elliptic } u(x).$$

More generally, a “finite-gap” potential for the Schrödinger operator is one for which the spectrum has the form

$$S = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \dots \cup [\lambda_{2G-2}, \lambda_{2G-1}] \cup [\lambda_{2G}, +\infty), \quad \text{for a finite-gap potential.}$$

The number of gaps is  $G$ . We will also see that the theory of such finite-gap potentials (and their dynamics under KdV) is related to the theory of hyperelliptic Riemann surfaces of genus  $G$ .

**Another look at the constant solutions of KdV.** Recall that for any constant  $\lambda_0$ , the constant function

$$u(x, t) \equiv -\lambda_0$$

is a solution of the KdV equation

$$u_t + uu_x + u_{xxx} = 0.$$

For this solution, the Lax pair equations, generally of the form

$$-6\phi_{xx} - u\phi = \lambda\phi, \quad \phi_t = -4\phi_{xxx} - u\phi_x - \frac{1}{2}u_x\phi,$$

look as follows:

$$-6\phi_{xx} = (\lambda - \lambda_0)\phi, \quad \phi_t = -4\phi_{xxx} + \lambda_0\phi_x.$$

There are two linearly independent simultaneous solutions,

$$\phi = \phi^\pm(x, t, \lambda) := \exp\left(\pm i\sqrt{\frac{\lambda - \lambda_0}{6}}\left[x + \frac{1}{3}(2\lambda - \lambda_0)t\right]\right).$$

These are perhaps best thought of as two branches of a single function  $\phi(x, t, P)$ , where  $P$  is a point on the *Riemann surface*  $\Gamma$  defined as the set of points  $(\lambda, y) \in \mathbb{C}^2$  that satisfy the equation

$$y^2 = \frac{1}{6}(\lambda - \lambda_0).$$

One way to think of  $\Gamma$  is as two copies of the complex  $\lambda$ -plane sewn together along the branch cut  $(-\infty, \lambda_0)$ . We have two natural functions on  $\Gamma$ ,  $\lambda(P)$ , which just “forgets” which sheet we are on and gives us back the corresponding  $\lambda$ -value, and  $y(P)$ , which contains the information about which sheet the point  $P$  is on. Of

course these two functions are not completely independent of each other, being linked for all  $P \in \Gamma$  by the equation above.

We cannot avoid Riemann surfaces (and coincident methods of algebraic geometry) if we are going to develop the theory of “multiphase wave” solutions of the KdV equation. But we can delay their appearance in the theory for awhile. To see how, notice that while  $\phi(x, t, P)$  is a function on the Riemann surface  $\Gamma$ , quadratic forms in  $\phi(x, t, P)$  are simple functions of  $\lambda$  alone. Indeed, if we define

$$\begin{aligned} a &:= \phi^+(x, t, \lambda)\phi^-(x, t, \lambda), \\ b &:= \phi_x^+(x, t, \lambda)\phi_x^-(x, t, \lambda), \\ c &:= \frac{1}{2} (\phi^+(x, t, \lambda)\phi_x^-(x, t, \lambda) + \phi_x^+(x, t, \lambda)\phi^-(x, t, \lambda)), \end{aligned}$$

and

$$W := \frac{1}{2i} (\phi^+(x, t, \lambda)\phi_x^-(x, t, \lambda) - \phi_x^+(x, t, \lambda)\phi^-(x, t, \lambda)),$$

then

$$\begin{aligned} a &= 1, \\ b &= \frac{1}{6}(\lambda - \lambda_0), \\ c &= 0, \end{aligned}$$

and

$$W^2 = \frac{1}{6}(\lambda - \lambda_0),$$

so that  $a$ ,  $b$ , and  $c$  are single-valued functions of  $\lambda$ . In fact, in this case they turned out to be polynomials in  $\lambda$ . This is a feature we can try to mimic more generally.

**The “squared eigenfunction” equations.** For each solution  $u(x, t)$  of KdV we can find two simultaneous solutions  $\phi^\pm(x, t, \lambda)$  of the Lax pair equations. The Lax pair equations are linear equations in  $\phi = \phi^\pm$ , so it is somewhat remarkable that we can derive linear equations satisfied by the quadratic forms  $a$ ,  $b$ ,  $c$ , and  $W$ . However, direct calculations bear this out. We can write our results in system form as follows:

$$\frac{\partial}{\partial x} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -\frac{1}{3}(u + \lambda) \\ -\frac{1}{6}(u + \lambda) & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and

$$\frac{\partial}{\partial t} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{3}u_x & 0 & \frac{4}{3}\lambda - \frac{2}{3}u \\ 0 & -\frac{1}{3}u_x & \frac{1}{9}[3u_{xx} + u^2 - \lambda u - 2\lambda^2] \\ \frac{1}{18}[u_{xx} + u^2 - \lambda u - 2\lambda^2] & \frac{2}{3}\lambda - \frac{1}{3}u & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

We call these systems the *squared eigenfunction equations*, and refer to  $a$ ,  $b$ , and  $c$  as *squared eigenfunctions*. Moreover, by direct calculation,

$$\frac{\partial W}{\partial x} = \frac{\partial W}{\partial t} = 0.$$

( $W$  is proportional to the Wronskian of  $\phi^+$  and  $\phi^-$ .) Even further,  $W$  can be related to  $a$ ,  $b$ , and  $c$  as follows:

$$c^2 = \frac{1}{4}(\phi^+)^2(\phi_x^-)^2 + \frac{1}{4}(\phi_x^+)^2(\phi^-)^2 + \frac{1}{2}\phi^+\phi^-\phi_x^+\phi_x^-$$

so

$$c^2 - ab = \frac{1}{4}(\phi^+)^2(\phi_x^-)^2 + \frac{1}{4}(\phi_x^+)^2(\phi^-)^2 - \frac{1}{2}\phi^+\phi^-\phi_x^+\phi_x^- = -W^2.$$

Therefore  $W^2 = ab - c^2$  is constant in  $x$  and  $t$  when  $a$ ,  $b$ , and  $c$  evolve according to the compatible squared eigenfunction equations.

**Polynomial in  $\lambda$  solutions of the squared eigenfunction equations.** Let  $G = 0, 1, 2, \dots$  be a fixed integer. We consider now the possibility that there exist solutions  $u = u(x, t)$  of KdV for which the squared eigenfunction equations admit simultaneous solutions of the form

$$\begin{aligned} a(x, t, \lambda) &= \lambda^G + a_{G-1}(x, t)\lambda^{G-1} + \dots + a_1(x, t)\lambda + a_0(x, t), \\ b(x, t, \lambda) &= \frac{1}{6}\lambda^{G+1} + b_G(x, t)\lambda^G + b_{G-1}(x, t)\lambda^{G-1} + \dots + b_1(x, t)\lambda + b_0(x, t), \end{aligned}$$

and

$$c(x, t, \lambda) = c_{G-1}(x, t)\lambda^{G-1} + \dots + c_1(x, t)\lambda + c_0(x, t).$$

The coefficients are all determined explicitly in terms of  $u$  and its derivatives, and therefore all three of these are real polynomials. For each such solution of KdV, the square of  $W$  is necessarily a polynomial in  $\lambda$  of degree  $2G + 1$  with leading coefficient  $1/6$ :

$$W^2 = ab - c^2 = \frac{1}{6}\lambda^{2G+1} + P_{2G}\lambda^{2G} + P_{2G-1}\lambda^{2G-1} + \dots + P_1\lambda + P_0.$$

The coefficients  $P_k$  are determined from the solution  $u(x, t)$  of KdV under consideration. However, we already know that  $W$  is independent of  $x$  and  $t$ . Therefore, the  $P_k$  are all independent of  $x$  and  $t$  and give constants of motion for this solution of KdV.

In view of the constancy of the  $P_k$ , there exist for this solution  $u(x, t)$  of KdV a set of  $2G + 1$  constant numbers that are the roots of the polynomial  $W^2$ . We assume that these roots are all real and are ordered:  $\lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_{2G-1} < \lambda_{2G}$ . Then,  $W^2$  can be written in the form

$$W^2 = \frac{1}{6} \prod_{n=0}^{2G} (\lambda - \lambda_n).$$

Note in passing that this equation defines a Riemann surface  $\Gamma$  that is a two-sheeted covering of the  $\lambda$ -plane. Such Riemann surfaces are called *hyperelliptic*. As  $a(x, t, \lambda)$  is a polynomial, we may also write down a representation of it in terms of its roots:

$$a(x, t, \lambda) = \prod_{k=1}^G (\lambda - \mu_k(x, t)),$$

where the coefficients  $a_g(x, t)$  are symmetric functions of the roots  $\mu_k(x, t)$ . Now, an interesting calculation is to evaluate both sides of the equation  $W^2 = ab - c^2$  for  $\lambda = \mu_k(x, t)$ . Then since these are the roots of  $a$ , we have

$$\frac{1}{6} \prod_{n=0}^{2G} (\mu_k(x, t) - \lambda_n) = -c(x, t, \mu_k(x, t))^2 \leq 0.$$

As the  $\lambda_n$  are all real, this equation forces all  $G$  roots  $\mu_k(x, t)$  to be real as well. Moreover, the negativity of the right-hand side tells us that the  $G$  roots  $\mu_k(x, t)$  have to lie in the union of open intervals

$$\mu_k(x, t) \in (-\infty, \lambda_0) \cup (\lambda_1, \lambda_2) \cup \dots \cup (\lambda_{2G-1}, \lambda_{2G}).$$

For the solutions of interest here, we will assume that the  $G$  roots  $\mu_k(x, t)$  are confined one to each of the finite subintervals. Therefore we will order the roots as  $\mu_1(x, t) < \mu_2(x, t) < \dots < \mu_G(x, t)$ , and then insist that

$$\lambda_{2k-1} \leq \mu_k(x, t) \leq \lambda_{2k}, \quad k = 1, \dots, G.$$

**A formula for  $u$  from its squared eigenfunction data.** For each solution  $u(x, t)$  of KdV that admits polynomial squared eigenfunctions of the type described above, we will now give a formula that represents  $u(x, t)$  in terms of the spectral data  $\{\lambda_n\}$  and  $\{\mu_k(x, t)\}$ . To obtain such a formula, let us look at the coefficient of  $\lambda^{2G}$  in the relation  $W^2 = ab - c^2$ : using the product representation of  $W^2$  to find the coefficient of  $\lambda^{2G}$  on the left-hand side, we get

$$-\frac{1}{6} \sum_{n=0}^{2G} \lambda_n = \frac{1}{6} a_{G-1}(x, t) + b_G(x, t).$$

Now, from the product representation of  $a(x, t, \lambda)$  we can easily express  $a_{G-1}(x, t)$  in terms of the roots:

$$a_{G-1}(x, t) = - \sum_{k=1}^G \mu_k(x, t),$$

so

$$- \sum_{n=0}^{2G} \lambda_n = - \frac{1}{6} \sum_{k=1}^G \mu_k(x, t) + b_G(x, t).$$

What is  $b_G(x, t)$ ? From the squared eigenfunction equation

$$c_x = - \frac{1}{6} (u + \lambda) a + b$$

we can substitute in the polynomial forms for  $a$ ,  $b$ , and  $c$ , and extract the coefficient of  $\lambda^G$ , which gives the relation

$$0 = - \frac{1}{6} u - \frac{1}{6} a_{G-1} + b_G,$$

so

$$b_G(x, t) = \frac{1}{6} (u + a_{G-1}) = \frac{1}{6} u(x, t) - \frac{1}{6} \sum_{k=1}^G \mu_k(x, t).$$

Therefore, we have

$$u(x, t) = 2 \sum_{k=1}^G \mu_k(x, t) - \sum_{n=0}^{2G} \lambda_n.$$

**Differential equations satisfied by the  $\mu_k(x, t)$ .** We now reverse our point of view. Rather than thinking of  $u$  as being given, we now consider the possibility of finding  $\{\mu_k(x, t)\}$  given some numbers  $\lambda_0 < \lambda_1 < \dots < \lambda_{2G}$ , and then using the above formula to reconstruct a solution  $u(x, t)$  of the KdV equation.

As the  $\mu_k(x, t)$  are the roots of  $a(x, t, \lambda)$  we can obtain equations governing the  $\mu_k(x, t)$  from the squared eigenfunction equations for  $a(x, t, \lambda)$ :

$$\frac{\partial a}{\partial x} = 2c, \quad \frac{\partial a}{\partial t} = \frac{1}{3} u_x a + \left( \frac{4}{3} \lambda - \frac{2}{3} u \right) c.$$

Now, using the product rule and the representation of  $a$  in terms of its roots, we see that

$$\frac{\partial a}{\partial x} = \frac{\partial}{\partial x} \prod_{k=1}^G (\lambda - \mu_k) = - \sum_{k=1}^G \frac{\partial \mu_k}{\partial x} \prod_{\substack{j=1 \\ j \neq k}}^G (\lambda - \mu_j),$$

and similarly,

$$\frac{\partial a}{\partial t} = - \sum_{k=1}^G \frac{\partial \mu_k}{\partial t} \prod_{\substack{j=1 \\ j \neq k}}^G (\lambda - \mu_j).$$

Now we substitute into the squared eigenfunction equations for  $a(x, t, \lambda)$ :

$$\begin{aligned} - \sum_{k=1}^G \frac{\partial \mu_k}{\partial x} \prod_{\substack{j=1 \\ j \neq k}}^G (\lambda - \mu_j) &= 2c(x, t, \lambda), \\ - \sum_{k=1}^G \frac{\partial \mu_k}{\partial t} \prod_{\substack{j=1 \\ j \neq k}}^G (\lambda - \mu_j) &= \frac{1}{3} u_x \prod_{j=1}^G (\lambda - \mu_j) + \left( \frac{4}{3} \lambda - \frac{2}{3} u \right) c(x, t, \lambda). \end{aligned}$$

As  $\lambda \in \mathbb{C}$  is a parameter in these equations, we can evaluate them for  $\lambda = \mu_k$ ,  $k = 1, \dots, G$ :

$$- \frac{\partial \mu_k}{\partial x} \prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j) = 2c(x, t, \mu_k), \quad - \frac{\partial \mu_k}{\partial t} \prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j) = \left( \frac{4}{3} \mu_k - \frac{2}{3} u \right) c(x, t, \mu_k),$$

for  $k = 1, \dots, G$ . Next, we eliminate  $u$  by using its representation in terms of  $\{\lambda_n\}$  and  $\{\mu_k(x, t)\}$ :

$$-\frac{\partial \mu_k}{\partial x} \prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j) = 2c(x, t, \mu_k), \quad -\frac{\partial \mu_k}{\partial t} \prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j) = \left( \frac{4}{3} \mu_k + \frac{2}{3} \sum_{n=0}^{2G} \lambda_n - \frac{4}{3} \sum_{j=1}^G \mu_j \right) c(x, t, \mu_k),$$

for  $k = 1, \dots, G$ . The final step in obtaining closed equations for the  $\{\mu_k(x, t)\}$  is to recall the fact that

$$c(x, t, \mu_k)^2 = -W(x, t, \mu_k)^2 = -\frac{1}{6} \prod_{n=0}^{2G} (\mu_k - \lambda_n),$$

so upon taking a square root,

$$c(x, t, \mu_k) = \pm \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu_k - \lambda_n)}.$$

Thus, we obtain two systems of coupled nonlinear ODEs for  $\{\mu_k(x, t)\}$ :

$$\begin{aligned} -\frac{\partial \mu_k}{\partial x} \prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j) &= \pm 2 \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu_k - \lambda_n)}, \\ -\frac{\partial \mu_k}{\partial t} \prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j) &= \pm \left( \frac{4}{3} \mu_k + \frac{2}{3} \sum_{n=0}^{2G} \lambda_n - \frac{4}{3} \sum_{j=1}^G \mu_j \right) \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu_k - \lambda_n)}. \end{aligned}$$

**Linearization of the equations for  $\{\mu_k(x, t)\}$ .** Despite their terrible appearance, these equations can be linearized exactly. The trick is to introduce the right combinations of the unknown variables. So, first rewrite the equations in the form

$$\begin{aligned} \frac{1}{\pm \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu_k - \lambda_n)}} \frac{\partial \mu_k}{\partial x} &= -2 \frac{1}{\prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j)}, \\ \frac{1}{\pm \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu_k - \lambda_n)}} \frac{\partial \mu_k}{\partial t} &= \frac{\frac{4}{3} \sum_{j=1}^G \mu_j - \frac{4}{3} \mu_k - \frac{2}{3} \sum_{n=0}^{2G} \lambda_n}{\prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j)}. \end{aligned}$$

Now, we will replace these equations by weighted sums as follows: multiply through by  $\mu_k^p$  for  $p = 0, 1, 2, \dots, G-1$ , and sum over  $k$ :

$$\begin{aligned} \sum_{k=1}^G \frac{\mu_k^p}{\pm \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu_k - \lambda_n)}} \frac{\partial \mu_k}{\partial x} &= -2 \sum_{k=1}^G \frac{\mu_k^p}{\prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j)}, \\ \sum_{k=1}^G \frac{\mu_k^p}{\pm \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu_k - \lambda_n)}} \frac{\partial \mu_k}{\partial t} &= \frac{2}{3} \sum_{k=1}^G \frac{2\mu_k^p \sum_{j=1}^G \mu_j - 2\mu_k^{p+1} - \mu_k^p \sum_{n=0}^{2G} \lambda_n}{\prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j)}. \end{aligned}$$

Now, the left-hand sides can be re-written as derivatives of some sums of integrals:

$$\begin{aligned} \frac{\partial}{\partial x} \sum_{k=1}^G \int_z^{\mu_k} \frac{\mu^p d\mu}{\pm \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu - \lambda_n)}} &= -2 \sum_{k=1}^G \frac{\mu_k^p}{\prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j)}, \\ \frac{\partial}{\partial t} \sum_{k=1}^G \int_z^{\mu_k} \frac{\mu^p d\mu}{\pm \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu - \lambda_n)}} &= \frac{2}{3} \sum_{k=1}^G \frac{2\mu_k^p \sum_{j=1}^G \mu_j - 2\mu_k^{p+1} - \mu_k^p \sum_{n=0}^{2G} \lambda_n}{\prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j)}. \end{aligned}$$

Now, the claim is that the right-hand sides are in fact independent of the  $\{\mu_k\}$  even though they look complicated. To prove this, for  $q = 0, 1, 2, \dots$ , let  $I_q$  denote the contour integral

$$I_q := \frac{1}{2\pi i} \oint_C \frac{z^q dz}{\prod_{j=1}^G (z - \mu_j)},$$

where  $C$  is a counterclockwise-oriented contour that encircles all of the points  $z = \mu_1, \dots, \mu_G$ . On one hand, we can evaluate this integral by residues at the finite poles of the integrand, which gives

$$I_q = \sum_{k=1}^G \frac{\mu_k^q}{\prod_{\substack{j=1 \\ j \neq k}}^G (\mu_k - \mu_j)}.$$

On the other hand, we can also evaluate  $I_q$  by a residue at infinity. Noting that as  $z \rightarrow \infty$ ,

$$\begin{aligned} \prod_{j=1}^G (z - \mu_j)^{-1} &= z^{-G} \prod_{j=1}^G \left(1 - \frac{\mu_j}{z}\right)^{-1} \\ &= z^{-G} \prod_{j=1}^G \left(1 + \frac{\mu_j}{z} + O(z^{-2})\right) \\ &= z^{-G} \left(1 + \frac{1}{z} \sum_{j=1}^G \mu_j + O(z^{-2})\right) \\ &= z^{-G} + z^{-G-1} \sum_{j=1}^G \mu_j + O(z^{-G-2}), \end{aligned}$$

we evaluate the residue at infinity to see that

$$I_q = 0, \quad \text{for } q - G < -1, \quad I_{G-1} = 1, \quad \text{and} \quad I_G = \sum_{j=1}^G \mu_j.$$

Equating the representations of  $I_q$  gives the desired identities that dramatically simplify the right-hand sides of our differential equations:

$$\frac{\partial}{\partial x} \sum_{k=1}^G \int_z^{\mu_k} \frac{\mu^p d\mu}{\pm \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu - \lambda_n)}} = \begin{cases} 0, & p = 0, 1, 2, \dots, G-2, \\ -2, & p = G-1. \end{cases}$$

and

$$\frac{\partial}{\partial t} \sum_{k=1}^G \int_z^{\mu_k} \frac{\mu^p d\mu}{\pm \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu - \lambda_n)}} = \begin{cases} 0, & p = 0, 1, 2, \dots, G-3, \\ -\frac{4}{3}, & p = G-2, \\ -\frac{2}{3} \sum_{n=0}^{2G} \lambda_n, & p = G-1. \end{cases}$$

The fact that what appears on the right-hand sides of these equations is independent of  $x$  and  $t$  proves the compatibility of these differential equations. Moreover, we may now integrate the equations explicitly:

$$S_p := \sum_{k=1}^G \int_z^{\mu_k} \frac{\mu^p d\mu}{\pm \sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu - \lambda_n)}} = \alpha_p x - \beta_p t - \xi_p,$$

where

$$\alpha_p := \begin{cases} 0, & p = 0, 1, 2, \dots, G-2, \\ -2, & p = G-1, \end{cases} \quad \beta_p := \begin{cases} 0, & p = 0, 1, 2, \dots, G-3, \\ \frac{4}{3}, & p = G-2, \\ \frac{2}{3} \sum_{n=0}^{2G} \lambda_n, & p = G-1, \end{cases}$$

and  $\xi_p$  is a constant of integration. It remains only to solve for  $\mu_1, \dots, \mu_G$  in terms of  $S_0, \dots, S_{G-1}$ .

**The Jacobi inversion problem.** The problem of finding  $\mu_1, \dots, \mu_G$  from  $S_0, \dots, S_{G-1}$  is a classical problem of algebraic geometry, called the *Jacobi inversion problem*.

*The case of  $G = 1$ . Elliptic functions.* If  $G = 1$ , then we have to determine  $\mu(x, t)$  from the single equation

$$\int_z^\mu \frac{d\mu}{\pm \sqrt{-(\mu - \lambda_0)(\mu - \lambda_1)(\mu - \lambda_2)}} = -\frac{2}{\sqrt{6}} \left( x - x_0 + \frac{1}{3}(\lambda_0 + \lambda_1 + \lambda_2)t \right),$$

where  $\lambda_0 < \lambda_1 < \mu < \lambda_2$ . Now this formula requires some interpretation, in view of the arbitrary  $\pm$  sign in the denominator and the fact that the right-hand side can become arbitrarily large as  $x$  and  $t$  vary, while the point  $\mu$  is supposed to remain bounded in a fixed interval. The idea is the following: first take the lower limit of integration  $z$  to be in the same interval as  $\mu$ . Then, for a given sign of the denominator, the integral will increase in absolute value as  $\mu$  moves away from  $z$ . This will continue until  $\mu$  hits the end of its interval. How can we make the integral continue to increase in value (as the right-hand side of the equation most certainly can)? Simply “switch onto the other sheet” by changing the sign of the denominator, and allowing  $\mu$  to return backwards in the direction from which it approached the endpoint. In this way, the point  $\mu$  can continue to oscillate back and forth between  $\lambda_1$  and  $\lambda_2$  while the integral increases in value without bound. We can also see from this argument that  $\mu(x, t)$  will be a periodic function of  $x$  and  $t$ .

To be more concrete about the idea, suppose instead we had to solve the equation

$$\int_z^\mu \frac{d\mu}{\pm \sqrt{1 - \mu^2}} = \alpha x - \beta t - \xi.$$

Here  $\mu$  should oscillate back and forth in the interval  $(-1, 1)$  as  $x$  and  $t$  vary, according to the preceding argument. But we can evaluate this integral in closed form:

$$\pm(\arcsin(\mu) - \arcsin(z)) = \alpha x - \beta t - \xi,$$

so we indeed obtain periodic motion in  $x$  and  $t$  by solving for  $\mu$ :

$$\mu = \sin(\pm(\alpha x - \beta t - \xi')).$$

The left-hand side we are faced with is an example of an *elliptic integral of the first kind*. Generally an elliptic integral of the first kind is simply one in which the denominator is the reciprocal of the square root of a cubic or quartic polynomial in  $\mu$ . We have the cubic case here. It can be transformed into a simpler form with the substitution

$$\mu = \lambda_2 - (\lambda_2 - \lambda_1) \sin^2(\phi).$$

With this substitution, we get

$$\begin{aligned} & \frac{d\mu}{\pm\sqrt{(\mu - \lambda_0)(\mu - \lambda_1)(\lambda_2 - \mu)}} \\ &= \frac{-2(\lambda_2 - \lambda_1) \sin(\phi) \cos(\phi) d\phi}{\pm\sqrt{(\lambda_2 - \lambda_0 - (\lambda_2 - \lambda_1) \sin^2(\phi))(\lambda_2 - \lambda_1 - (\lambda_2 - \lambda_1) \sin^2(\phi))((\lambda_2 - \lambda_1) \sin^2(\phi))}} \\ &= \frac{-2 d\phi}{\pm\sqrt{\lambda_2 - \lambda_0 - (\lambda_2 - \lambda_1) \sin^2(\phi)}}. \end{aligned}$$

Therefore, the problem we have to solve for  $G = 1$  is

$$\mp \frac{2}{\sqrt{\lambda_2 - \lambda_0}} \int_0^\phi \frac{d\phi}{\sqrt{1 - m \sin^2(\phi)}} = -\frac{2}{\sqrt{6}} \left( x - x_0 + \frac{1}{3}(\lambda_0 + \lambda_1 + \lambda_2)t \right),$$

where

$$m = \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_0}$$

which satisfies  $0 < m < 1$ , or

$$\int_0^\phi \frac{d\phi}{\sqrt{1 - m \sin^2(\phi)}} = \pm \sqrt{\frac{\lambda_2 - \lambda_0}{6} - \frac{\lambda_0}{6}} \left( x - x_0 + \frac{1}{3}(\lambda_0 + \lambda_1 + \lambda_2)t \right).$$

We have taken the lower limit to be zero because the  $x_0$  on the right-hand side is arbitrary anyway. Note that we can also write  $\mu$  in terms of  $\cos(\phi)$  by

$$\mu = \lambda_1 + (\lambda_2 - \lambda_1) \cos^2(\phi),$$

and by definition if the equation

$$\int_0^\phi \frac{d\phi}{\sqrt{1 - m \sin^2(\phi)}} = v,$$

is solved for  $\phi = \phi(v; m)$ , then

$$\text{cn}(v; m) := \cos(\phi(v; m)).$$

The function  $\text{cn}(v; m)$  is a *Jacobi elliptic function*. It is periodic in  $v$  with a period that is whatever it takes to increase  $\phi(v; m)$  by  $2\pi$ . Clearly, this period is

$$\Delta v(m) = \int_0^{2\pi} \frac{d\phi}{\sqrt{1 - m \sin^2(\phi)}}.$$

The wave form of  $\text{cn}(v; m)$  is not exactly sinusoidal, but depends on  $m$ . Some interesting limits are

$$\lim_{m \downarrow 0} \text{cn}(v; m) = \cos(v), \quad \lim_{m \uparrow 1} \text{cn}(v; m) = \text{sech}(v).$$

The solution of KdV that we have obtained for  $G = 1$  is

$$\begin{aligned} u(x, t) &= 2\mu(x, t) - \lambda_0 - \lambda_1 - \lambda_2 \\ &= 2\lambda_1 + 2(\lambda_2 - \lambda_1) \text{cn}^2 \left( \sqrt{\frac{\lambda_2}{6} - \frac{\lambda_0}{6}} \left( x - x_0 + \frac{1}{3}(\lambda_0 + \lambda_1 + \lambda_2)t \right); \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_0} \right) - \lambda_0 - \lambda_1 - \lambda_2 \\ &= \lambda_1 - \lambda_0 - \lambda_2 + 2(\lambda_2 - \lambda_1) \text{cn}^2 \left( \sqrt{\frac{\lambda_2}{6} - \frac{\lambda_0}{6}} \left( x - x_0 + \frac{1}{3}(\lambda_0 + \lambda_1 + \lambda_2)t \right); \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_0} \right). \end{aligned}$$

Consider the limit  $\lambda_1 \downarrow \lambda_0$ . This implies  $m \uparrow 1$ , so

$$\lim_{\lambda_1 \downarrow \lambda_0} u(x, t) = -\lambda_2 + 2(\lambda_2 - \lambda_0) \text{sech}^2 \left( \sqrt{\frac{\lambda_2}{6} - \frac{\lambda_0}{6}} \left( x - x_0 + \left( \frac{2\lambda_0}{3} + \frac{\lambda_2}{3} \right) t \right) \right).$$

We can obtain a solution decaying to zero for large  $x$  and  $t$  by choosing  $\lambda_2 = 0$ . This forces  $\lambda_0 < 0$  and we write

$$\lambda_0 = -\frac{3}{2}c$$

for  $c > 0$ . The limiting formula becomes

$$\lim_{\lambda_1 \downarrow \lambda_0 = -3c/2, \lambda_2 = 0} u(x, t) = 3c \operatorname{sech}^2 \left( \frac{\sqrt{c}}{2}(x - x_0 - ct) \right),$$

the soliton solution of KdV. More generally, for  $\lambda_1 > \lambda_0$  we have a periodic traveling wave solution of KdV with speed equal to the average value of the  $\{\lambda_n\}$ .

*The general case of  $G > 1$ . The Its-Matveev formula.* To begin to study the case of  $G > 1$ , we have to rearrange our equations somewhat by replacing  $S_0, \dots, S_{G-1}$  by certain linear combinations of them. The main difficulty we are faced with is that the quantities  $\{S_p\}$  are not well-defined given the  $\{\mu_k\}$  because the integrals that appear are not completely independent of path. To deal with this we need to come back to the hyperelliptic Riemann surface  $\Gamma$  defined by the relation

$$y^2 = -\frac{1}{6} \prod_{n=0}^{2G} (\lambda - \lambda_n).$$

One way to think of  $\Gamma$  is as two copies of the complex  $\lambda$ -plane sewn together along the branch cuts obtained by considering where  $y^2 < 0$ . The branch cuts are therefore the intervals

$$I = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \dots \cup [\lambda_{2G}, +\infty).$$

The variables  $\mu_k$  are confined to the complement of this set on the real axis, which allows them to lie on either of the two sheets. One should notice here the similarity of this branch cut structure with the finite-gap spectrum of a periodic Schrödinger operator.

The reason that the integrals defining the  $\{S_p\}$  are not independent of path is because on the Riemann surface  $\Gamma$  there exist closed paths that cannot be deformed to points. This is quite a different situation from integration on the complex  $\lambda$ -plane. Indeed, by ‘‘compactifying’’  $\Gamma$  (say by identifying the two copies of the complex  $\lambda$ -plane each with a copy of the Riemann sphere by stereographic projection from the north pole as in Figure 1 prior to sewing along the branch cuts as shown in Figure 2), and by smoothing out the surface

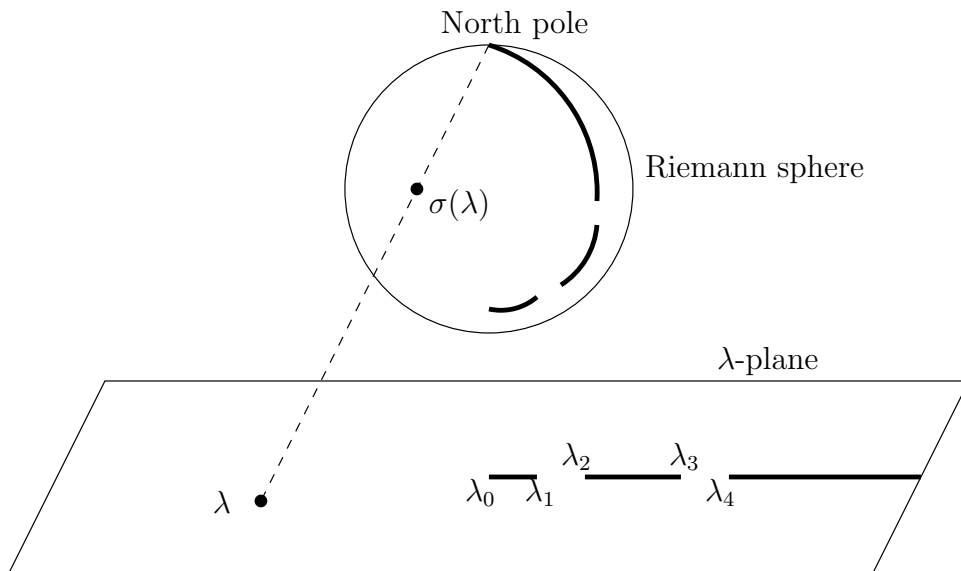


FIGURE 1. *The stereographic projection  $\sigma$  of the  $\lambda$ -plane onto the Riemann sphere.*

near the branch points (say by introducing appropriate analytic coordinates  $\tau_n = \sqrt{\lambda - \lambda_n}$  near  $\lambda = \lambda_n$  and

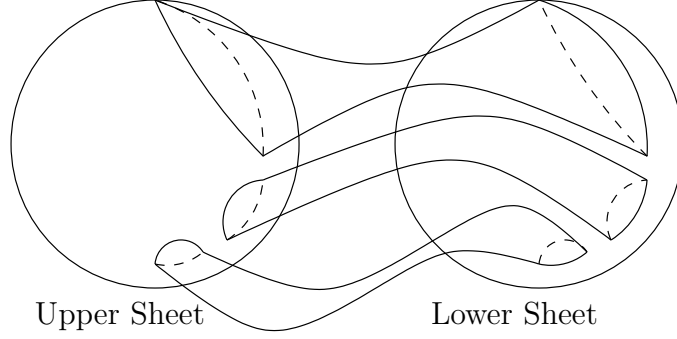


FIGURE 2. *The identification of two copies of the Riemann sphere along branch cuts.*

$\tau_\infty = 1/(-\lambda)^{1/2}$  near  $\lambda = \infty$ ), we obtain a model of  $\Gamma$  that is topologically a sphere with  $G$  handles. Such a surface is said to have *genus* equal to  $G$ . See Figure 3.

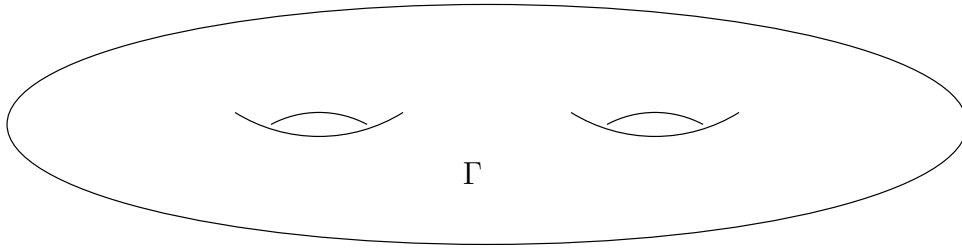


FIGURE 3. *A model of  $\Gamma$  as a complex manifold of genus  $G$  (shown here for  $G = 2$ ).*

To quantify the multivaluedness of the  $\{S_p\}$ , we therefore need to introduce a complete set of all essentially distinct (from the point of view of integration) closed contours on  $\Gamma$ . Such a set of closed contours, given proper orientation, is called a *homology basis* for  $\Gamma$ . A homology basis consists of  $2G$  oriented, closed, noncontractible loops on  $\Gamma$  (called *homology cycles*), and the basis should be independent in the sense that no one of the cycles can be obtained from combining the others and smoothly deforming. Furthermore, we should choose the basis so that it splits into two subsets of cycles,  $\{a_1, \dots, a_G\}$  and  $\{b_1, \dots, b_G\}$  such that the  $\{a_j\}$  are mutually nonintersecting as are the  $\{b_j\}$ , and such that in fact the only allowed intersections are between  $a_n$  and  $b_n$ , where we allow exactly one intersection, with directions chosen so that  $b_n$  crosses  $a_n$  once from the right of  $a_n$ . Such a homology basis is illustrated in Figure 4 for  $G = 2$ . This homology basis

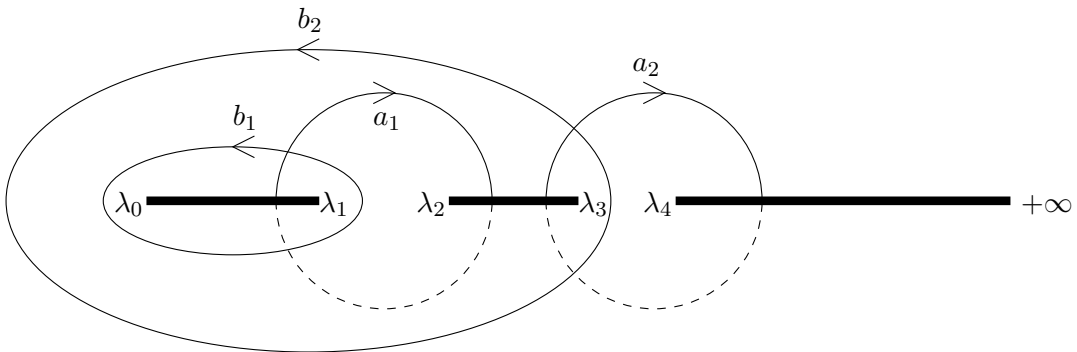


FIGURE 4. *The homology cycles for  $G = 2$ . Solid lines are the branch cuts, and dashed curves are continuations of cycles on the second sheet.*

has the extra property that for each  $j$  the cycle  $a_j$  is topologically equivalent to the path that is taken by the variable  $\mu_j$  as it varies in  $x$  and  $t$ .

With this machinery in place, we may now explain how to assemble the correct linear combinations of the  $\{S_p\}$ . Let's agree that the function  $y : \Gamma \rightarrow \mathbb{C}$  is real and positive on the sheet of the complex plane illustrated in Figure 4 for  $\lambda < \lambda_0$ . Then, we choose  $G$  polynomials  $H_1(\lambda), H_2(\lambda), \dots, H_G(\lambda)$  all of degree  $G - 1$  such that:

$$\oint_{a_j} \frac{H_k(\lambda(P))}{y(P)} d\lambda(P) = 2\pi i \delta_{jk}, \quad j, k = 1, \dots, G.$$

If we express the polynomials in terms of the coefficients

$$H_k(\lambda) = ih_k^{(G-1)} \lambda^{G-1} + ih_k^{(G-2)} \lambda^{G-2} + \dots + ih_k^{(1)} \lambda + ih_k^{(0)},$$

then the coefficients are determined by solving the system

$$\begin{bmatrix} \oint_{a_1} \frac{d\lambda(P)}{y(P)} & \oint_{a_1} \frac{\lambda(P) d\lambda(P)}{y(P)} & \dots & \oint_{a_1} \frac{\lambda(P)^{G-1} d\lambda(P)}{y(P)} \\ \vdots & \vdots & & \vdots \\ \oint_{a_G} \frac{d\lambda(P)}{y(P)} & \oint_{a_G} \frac{\lambda(P) d\lambda(P)}{y(P)} & \dots & \oint_{a_G} \frac{\lambda(P)^{G-1} d\lambda(P)}{y(P)} \end{bmatrix} \begin{bmatrix} h_1^{(0)} & h_2^{(0)} & \dots & h_G^{(0)} \\ \vdots & \vdots & & \vdots \\ h_1^{(G-1)} & h_2^{(G-1)} & \dots & h_G^{(G-1)} \end{bmatrix} = 2\pi \mathbb{I}.$$

Note also that

$$\oint_{a_k} \frac{\lambda(P)^p d\lambda(P)}{y(P)} = 2(-1)^k \int_{\lambda_{2k-1}}^{\lambda_{2k}} \frac{\lambda^p d\lambda}{\sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\lambda - \lambda_n)}},$$

where now we mean the positive square root. Therefore, the coefficient matrix is

$$\begin{bmatrix} h_1^{(0)} & h_2^{(0)} & \dots & h_G^{(0)} \\ \vdots & \vdots & & \vdots \\ h_1^{(G-1)} & h_2^{(G-1)} & \dots & h_G^{(G-1)} \end{bmatrix} = 2\pi \begin{bmatrix} \oint_{a_1} \frac{d\lambda(P)}{y(P)} & \oint_{a_1} \frac{\lambda(P) d\lambda(P)}{y(P)} & \dots & \oint_{a_1} \frac{\lambda(P)^{G-1} d\lambda(P)}{y(P)} \\ \vdots & \vdots & & \vdots \\ \oint_{a_G} \frac{d\lambda(P)}{y(P)} & \oint_{a_G} \frac{\lambda(P) d\lambda(P)}{y(P)} & \dots & \oint_{a_G} \frac{\lambda(P)^{G-1} d\lambda(P)}{y(P)} \end{bmatrix}^{-1},$$

and we should now check that this is indeed an invertible matrix. But by multilinearity of the determinant,

$$\begin{aligned} \det \begin{bmatrix} \oint_{a_1} \frac{d\lambda(P)}{y(P)} & \oint_{a_1} \frac{\lambda(P) d\lambda(P)}{y(P)} & \dots & \oint_{a_1} \frac{\lambda(P)^{G-1} d\lambda(P)}{y(P)} \\ \vdots & \vdots & & \vdots \\ \oint_{a_G} \frac{d\lambda(P)}{y(P)} & \oint_{a_G} \frac{\lambda(P) d\lambda(P)}{y(P)} & \dots & \oint_{a_G} \frac{\lambda(P)^{G-1} d\lambda(P)}{y(P)} \end{bmatrix} \\ = \oint_{a_1} \frac{d\lambda(P_1)}{y(P_1)} \oint_{a_2} \frac{d\lambda(P_2)}{y(P_2)} \dots \oint_{a_G} \frac{d\lambda(P_G)}{y(P_G)} \det \begin{bmatrix} 1 & \lambda(P_1) & \dots & \lambda(P_1)^{G-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda(P_G) & \dots & \lambda(P_G)^{G-1} \end{bmatrix} \\ = 2^G (-1)^{G(G+1)/2} \int_{\lambda_1}^{\lambda_2} \frac{d\mu_1}{\sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu_1 - \lambda_n)}} \dots \int_{\lambda_{2G-1}}^{\lambda_{2G}} \frac{d\mu_G}{\sqrt{-\frac{1}{6} \prod_{n=0}^{2G} (\mu_G - \lambda_n)}} \det \begin{bmatrix} 1 & \mu_1 & \dots & \mu_1^{G-1} \\ \vdots & \vdots & & \vdots \\ 1 & \mu_G & \dots & \mu_G^{G-1} \end{bmatrix}. \end{aligned}$$

The final determinant in the integrand is of Vandermonde type. We have

$$\det \begin{bmatrix} 1 & \mu_1 & \dots & \mu_1^{G-1} \\ \vdots & \vdots & & \vdots \\ 1 & \mu_G & \dots & \mu_G^{G-1} \end{bmatrix} = \prod_{k=1}^G \prod_{j=1}^{k-1} (\mu_k - \mu_j).$$

As the integrand of our multiple-integral formula for the determinant is always positive, so is the multiple integral, and indeed we can solve uniquely for the coefficients of the polynomials  $H_k(\lambda)$ . The numbers  $h_k^{(n)}$  are evidently all real numbers. We say that the differentials

$$\omega_k(P) := H_k(\lambda(P)) d\lambda(P)/y(P)$$

form a *normalized basis for the space of holomorphic differentials* on  $\Gamma$ . The coefficients of the polynomials  $H_k(\lambda)$  now define for us the correct linear combinations of the  $\{S_p\}$  for us to consider:

$$\begin{aligned} T_k &:= ih_k^{(G-1)} S_{G-1} + ih_k^{(G-2)} S_{G-2} + \cdots + ih_k^{(1)} S_1 + ih_k^{(0)} S_0 \\ &= \sum_{j=1}^G \int_{P_0}^{P_j} \frac{H_k(\lambda(P)) d\lambda(P)}{y(P)} \\ &= \sum_{j=1}^G \int_{P_0}^{P_j} \omega_k. \end{aligned}$$

Here  $P_0$  is an arbitrary base point on  $\Gamma$  and  $\mu_j = \lambda(P_j)$  for  $j = 1, \dots, G$ . Also, we take all paths from  $P_0$  to the same point  $P_j$  as being independent of  $k$ .

Now this basis is normalized on the  $\{a_j\}$  cycles, but the integrals on the  $\{b_j\}$  cycles will be nontrivial. Let  $\mathbf{B}$  be the  $G \times G$  matrix with entries

$$B_{jk} := \oint_{b_j} \omega_k.$$

It can be shown (see, for example, Farkas and Kra, *Riemann Surfaces*, Springer, 1992, section III.3) that  $\mathbf{B}$  is a complex-valued symmetric matrix ( $B_{jk} = B_{kj}$ ) and the real part of  $\mathbf{B}$  is negative definite:

$$\Re(\mathbf{v}^T \mathbf{B} \mathbf{v}) < 0, \quad \text{for all nonzero } \mathbf{v} \text{ in } \mathbb{R}^G.$$

(These statements hold for general homology bases on general Riemann surfaces. In our case, with a homology basis as shown in Figure 4, the matrix  $\mathbf{B}$  is real-symmetric and negative definite.)

Given a set of points  $P_1, \dots, P_G$  lying on the Riemann surface  $\Gamma$ , the value of the vector  $(T_1, T_2, \dots, T_G)^T$  is only defined modulo adding on to each of the integration paths various closed loops, which can be thought of as linear combinations of the homology cycles with integer coefficients. Therefore we should really be thinking of  $(T_1, T_2, \dots, T_G)^T$  as being an element of a “quotient space”:

$$\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_G \end{bmatrix} \in \mathbb{C}^G \quad \text{modulo} \quad 2\pi i n_1 \mathbf{e}_1 + 2\pi i n_2 \mathbf{e}_2 + \cdots + 2\pi i n_G \mathbf{e}_G + m_1 \mathbf{b}_1 + m_2 \mathbf{b}_2 + \cdots + m_G \mathbf{b}_G =: \mathbb{C}^G / \Lambda$$

where the  $n_k$  and  $m_k$  are arbitrary integers and the  $\mathbf{b}_k$  are the columns of the matrix  $\mathbf{B}$ . The part involving the unit vector  $\mathbf{e}_k$  comes from adding to the various integrals  $n_k$  copies of the  $a_k$  cycle and the part involving the vector  $\mathbf{b}_k$  comes from adding to the various integrals  $m_k$  copies of the  $b_k$  cycle. The integer combinations of the vectors  $2\pi i \mathbf{e}_k$  and  $\mathbf{b}_k$  make up a lattice in  $\mathbb{C}^G$  called  $\Lambda$ . The target space  $\mathbb{C}^G / \Lambda$  is therefore a  $G$ -complex dimensional torus associated with the Riemann surface  $\Gamma$ , called its *Jacobian variety* or for short, its *Jacobian*, denoted  $\text{Jac}(\Gamma)$ . The image point of our mapping lives in the Jacobian, and has the form

$$i \begin{bmatrix} \left( h_1^{(1)} \alpha_1 + \cdots + h_1^{(G-1)} \alpha_{G-1} \right) x - \left( h_1^{(1)} \beta_1 + \cdots + h_1^{(G-1)} \beta_{G-1} \right) t - \left( h_1^{(1)} \xi_1 + \cdots + h_1^{(G-1)} \xi_{G-1} \right) \\ \vdots \\ \left( h_G^{(1)} \alpha_1 + \cdots + h_G^{(G-1)} \alpha_{G-1} \right) x - \left( h_G^{(1)} \beta_1 + \cdots + h_G^{(G-1)} \beta_{G-1} \right) t - \left( h_G^{(1)} \xi_1 + \cdots + h_G^{(G-1)} \xi_{G-1} \right) \end{bmatrix}$$

modulo  $\Lambda$ . Note that since all of the  $\alpha_n$  are zero except for  $\alpha_{G-1}$  and all of the  $\beta_n$  are zero except for  $\beta_{G-2}$  and  $\beta_{G-1}$ , and since all of the  $\xi_n$  are arbitrary real constants, the image point in the Jacobian can be rewritten in simpler form:

$$i \begin{bmatrix} -2h_1^{(G-1)} \\ \vdots \\ -2h_G^{(G-1)} \end{bmatrix} x - i \begin{bmatrix} \frac{4}{3}h_1^{(G-2)} + \frac{2}{3}h_1^{(G-1)} \sum_{n=0}^{2G} \lambda_n \\ \vdots \\ \frac{4}{3}h_G^{(G-2)} + \frac{2}{3}h_G^{(G-1)} \sum_{n=0}^{2G} \lambda_n \end{bmatrix} t - i \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_G \end{bmatrix},$$

modulo  $\Lambda$ , where the  $\zeta_n$  are arbitrary real constants. Given a base point  $P_0 \in \Gamma$ , the mapping  $\mathbf{j} : \Gamma \rightarrow \text{Jac}(\Gamma)$  defined by the vector of integrals

$$\mathbf{j}(P) := \begin{bmatrix} \int_{P_0}^P \omega_1 \\ \int_{P_0}^P \omega_2 \\ \vdots \\ \int_{P_0}^P \omega_G \end{bmatrix}$$

is called the *Abel mapping*. Note that we are taking the path of integration to be the same in all components, but otherwise arbitrary; the fact that we could modify this path by adding on an  $a$ - or  $b$ -cycle is what makes  $\mathbf{j}(P)$  lie in the Jacobian rather than  $\mathbb{C}^G$ . With this notation, we see that our equations to determine the  $\{\mu_k\}$  can be written in the form

$$\mathbf{j}(P_1) + \mathbf{j}(P_2) + \cdots + \mathbf{j}(P_G) = i \begin{bmatrix} -2h_1^{(G-1)} \\ \vdots \\ -2h_G^{(G-1)} \end{bmatrix} x - i \begin{bmatrix} \frac{4}{3}h_1^{(G-2)} + \frac{2}{3}h_1^{(G-1)} \sum_{n=0}^{2G} \lambda_n \\ \vdots \\ \frac{4}{3}h_G^{(G-2)} + \frac{2}{3}h_G^{(G-1)} \sum_{n=0}^{2G} \lambda_n \end{bmatrix} t - i \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_G \end{bmatrix},$$

modulo  $\Lambda$ , and  $\mu_k = \lambda(P_k)$ .

We may now address the question of how the map from  $(P_1, \dots, P_G)$  to the Jacobian is inverted, at least modulo the ordering of the points  $P_j$ . The key player is the *Riemann theta function* of the surface  $\Gamma$ , which is defined as a Fourier series:

$$\theta(\mathbf{w}; \mathbf{B}) := \sum_{\mathbf{n} \in \mathbb{Z}^G} e^{\frac{1}{2}\mathbf{n}^T \mathbf{B} \mathbf{n}} e^{\mathbf{n}^T \mathbf{w}}, \quad \mathbf{w} \in \mathbb{C}^G.$$

This is a rapidly convergent series because the real part of  $\mathbf{B}$  is negative definite. The exponential rate of convergence is sufficient to make  $\theta(\mathbf{w}; \mathbf{B})$  an entire analytic function of  $\mathbf{w} \in \mathbb{C}^G$ . The theta function has some simple properties, as the following proposition shows.

**Proposition 1.** *The theta function of  $\Gamma$  is a so-called automorphic function:*

$$\theta(\mathbf{w} + 2\pi i \mathbf{e}_k; \mathbf{B}) = \theta(\mathbf{w}; \mathbf{B}) \quad \text{and} \quad \theta(\mathbf{w} + \mathbf{b}_k; \mathbf{B}) = e^{-\frac{1}{2}B_{kk}} e^{-w_k} \theta(\mathbf{w}; \mathbf{B})$$

for all  $k = 1, \dots, G$ .

*Proof.* The  $2\pi$ -periodicity in the imaginary coordinate directions  $i\mathbf{e}_k$  obviously holds term-by-term. The second statement has a short proof:

$$\theta(\mathbf{w} + \mathbf{b}_k; \mathbf{B}) := \sum_{\mathbf{n} \in \mathbb{Z}^G} e^{\frac{1}{2}\mathbf{n}^T \mathbf{B} \mathbf{n}} e^{\mathbf{n}^T \mathbf{w}} e^{\mathbf{n}^T \mathbf{b}_k}.$$

Now, replacing  $\mathbf{n}$  with  $\mathbf{m} - \mathbf{e}_k$ , where  $\mathbf{m}$  also ranges over  $\mathbb{Z}^G$ , we have

$$\begin{aligned} \theta(\mathbf{w} + \mathbf{b}_k; \mathbf{B}) &= \sum_{\mathbf{m} \in \mathbb{Z}^G} \left[ e^{\frac{1}{2}\mathbf{m}^T \mathbf{B} \mathbf{m}} e^{-\frac{1}{2}\mathbf{e}_k^T \mathbf{B} \mathbf{m}} e^{-\frac{1}{2}\mathbf{m}^T \mathbf{B} \mathbf{e}_k} e^{\frac{1}{2}\mathbf{e}_k^T \mathbf{B} \mathbf{e}_k} \right] \left[ e^{\mathbf{m}^T \mathbf{w}} e^{-\mathbf{e}_k^T \mathbf{w}} \right] \left[ e^{\mathbf{m}^T \mathbf{b}_k} e^{-\mathbf{e}_k^T \mathbf{b}_k} \right] \\ &= e^{\frac{1}{2}\mathbf{e}_k^T \mathbf{B} \mathbf{e}_k} e^{-\mathbf{e}_k^T \mathbf{b}_k} e^{-\mathbf{e}_k^T \mathbf{w}} \sum_{\mathbf{m} \in \mathbb{Z}^G} e^{\frac{1}{2}\mathbf{m}^T \mathbf{B} \mathbf{m}} e^{-\frac{1}{2}\mathbf{e}_k^T \mathbf{B} \mathbf{m}} e^{-\frac{1}{2}\mathbf{m}^T \mathbf{B} \mathbf{e}_k} e^{\mathbf{m}^T \mathbf{b}_k} e^{\mathbf{m}^T \mathbf{w}} \\ &= e^{\frac{1}{2}\mathbf{e}_k^T \mathbf{B} \mathbf{e}_k} e^{-\mathbf{e}_k^T \mathbf{b}_k} e^{-\mathbf{e}_k^T \mathbf{w}} \sum_{\mathbf{m} \in \mathbb{Z}^G} e^{\frac{1}{2}\mathbf{m}^T \mathbf{B} \mathbf{m}} e^{\mathbf{m}^T \mathbf{w}} \\ &= e^{-\frac{1}{2}B_{kk}} e^{-w_k} \sum_{\mathbf{m} \in \mathbb{Z}^G} e^{\frac{1}{2}\mathbf{m}^T \mathbf{B} \mathbf{m}} e^{\mathbf{m}^T \mathbf{w}}. \end{aligned}$$

The proof is complete upon recognizing the final series as  $\theta(\mathbf{w}; \mathbf{B})$ . □

Now consider a function of  $P \in \Gamma$  obtained by composing the theta function with the Abel mapping:

$$F(P; \mathbf{v}) := \theta(\mathbf{j}(P) - \mathbf{v}; \mathbf{B}).$$

Here  $\mathbf{v}$  is a  $\mathbb{C}^G$ -valued parameter vector. This function is not well-defined on  $\Gamma$  because the value of  $\mathbf{j}(P)$  is only defined modulo integer linear combinations of the lattice vectors  $\Lambda$ . However, we see from Proposition 1

that the indeterminacy in  $F$  amounts to multiplication by nonzero factors. Therefore it is fair to say that the zeros of  $F(P; \mathbf{v})$  are well-defined on  $\Gamma$ , even though  $F$  itself is not. The result we need is the following:

**Proposition 2.** *There is a constant vector  $\mathbf{k}$  depending only on  $\Gamma$  such when*

$$\mathbf{v} := \mathbf{j}(P_1) + \mathbf{j}(P_2) + \cdots + \mathbf{j}(P_G) + \mathbf{k},$$

*either the function  $F(P; \mathbf{v})$  vanishes identically, or it vanishes precisely at the points  $P = P_1, P_2, \dots, P_G$  (and nowhere else on  $\Gamma$ ), and the roots are all simple if the points are distinct.*

We will not prove this proposition here. The interested student should again consult Farkas and Kra. The vector  $\mathbf{k}$  is frequently called the vector of *Riemann constants*, and various formulae for it can be found in Farkas and Kra. It should be said that the case when  $F$  vanishes identically should be considered to be rather unusual, and moreover it turns out to *never happen* for the points  $P_k$  of interest in the KdV problem.

Note also that for the points of interest in the KdV problem we *know* the value of the vector  $\mathbf{v}$ :

$$\mathbf{v} = \mathbf{v}(x, t) := i \begin{bmatrix} -2h_1^{(G-1)} \\ \vdots \\ -2h_G^{(G-1)} \end{bmatrix} x - i \begin{bmatrix} \frac{4}{3}h_1^{(G-2)} + \frac{2}{3}h_1^{(G-1)} \sum_{n=0}^{2G} \lambda_n \\ \vdots \\ \frac{4}{3}h_G^{(G-2)} + \frac{2}{3}h_G^{(G-1)} \sum_{n=0}^{2G} \lambda_n \end{bmatrix} t - i \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_G \end{bmatrix} + \mathbf{k}.$$

So what we need to know are the zeros of  $F(P; \mathbf{v}(x, t))$ , or more properly the sum of their  $\lambda$ -values, since these are what occur as the sum of the  $\{\mu_k(x, t)\}$  in the solution formula

$$u(x, t) = 2 \sum_{k=1}^G \mu_k(x, t) - \sum_{n=0}^{2G} \lambda_n = 2 \sum_{k=1}^G \lambda(P_k(x, t)) - \sum_{n=0}^{2G} \lambda_n.$$

The idea to use is that of the *canonical dissection*  $\tilde{\Gamma}$  of the Riemann surface  $\Gamma$ . If we deform the homology cycles of  $\Gamma$  so that they all meet at a common point  $Q \in \Gamma$ , and then cut the Riemann surface along all of these cycles, it turns out to open up into a single polygon  $\tilde{\Gamma}$  with  $4G$  sides. An appropriate deformation of the homology cycles from Figure 4 is illustrated in Figure 5. A polygon (topologically speaking)  $\tilde{\Gamma}$  appears

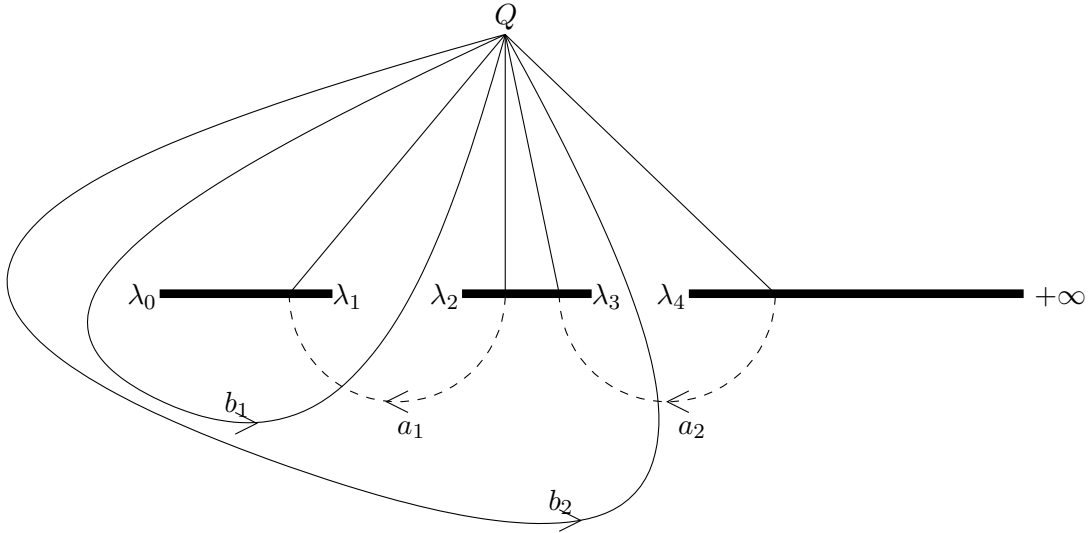


FIGURE 5. *Deforming the cycles on  $\Gamma$  to prepare for a canonical dissection.*

on the left as we traverse the contours  $a_1$ , then  $b_1$ , then  $a_1$  in reverse order, then  $b_1$  in reverse order, and then similarly for  $a_2$  and  $b_2$ . The polygon  $\tilde{\Gamma}$  corresponding to cutting  $\Gamma$  along the contours shown in Figure 5 in this way is shown in Figure 6. Note that each vertex of  $\tilde{\Gamma}$  corresponds to the same point  $Q$  of  $\Gamma$ .

To use the canonical dissection  $\tilde{\Gamma}$  to solve our Jacobi inversion problem, consider the contour integral

$$J := \frac{1}{2\pi i} \oint_{\partial \tilde{\Gamma}} \lambda(P) d \log F(P; \mathbf{v}).$$

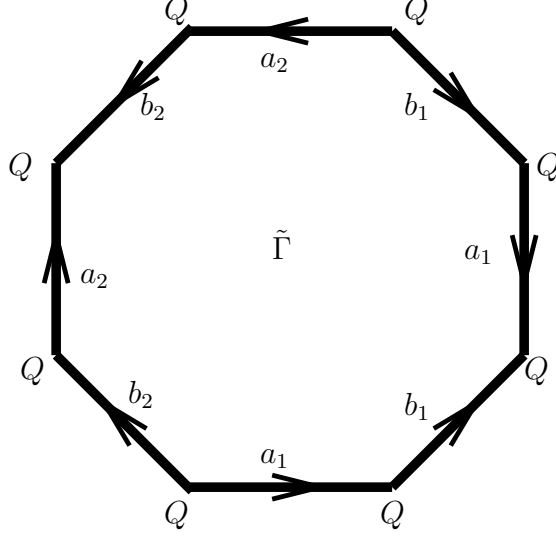


FIGURE 6. The canonical dissection  $\tilde{\Gamma}$  of the Riemann surface  $\Gamma$  for  $G = 2$ .

Here the boundary  $\partial\tilde{\Gamma}$  of the polygon  $\tilde{\Gamma}$  is considered to be oriented in the counterclockwise direction. We can evaluate  $J$  in two ways:

- On the one hand, we can calculate  $J$  by residues; the only singularities of the integrand are simple poles at the points  $P_1, \dots, P_G$  where  $F(P; \mathbf{v})$  has simple zeros for  $\mathbf{v} = \mathbf{j}(P_1) + \dots + \mathbf{j}(P_G) + \mathbf{k}$ , and the point in  $\tilde{\Gamma}$  corresponding to  $\lambda = \infty$ .
- On the other hand, we can calculate  $J$  by integration along the boundary  $\partial\tilde{\Gamma}$ , making use of the automorphic properties of the theta function.

By equating these two ways of evaluating the same thing, we will arrive at a formula for the sum of the  $\{\mu_k\}$ . First we evaluate the integral directly using the automorphic properties of the theta function. Obviously the integral can be written as a sum

$$J = \frac{1}{2\pi i} \sum_{k=1}^G \left[ \int_{a_k} \lambda(P) d \log F(P; \mathbf{v}) + \int_{b_k} \lambda(P) d \log F(P; \mathbf{v}) - \int_{a_k \text{ (after } b_k)} \lambda(P) d \log F(P; \mathbf{v}) - \int_{b_k \text{ (after } a_k)} \lambda(P) d \log F(P; \mathbf{v}) \right].$$

Note that the terms in the sum do not cancel because by the automorphic properties of the theta function, the integrands in the integrals on the second line are not the same as those in the integrals on the first line as  $P$  has traversed a cycle and so the argument of the theta function has been changed by  $\mathbf{b}_k$  or  $2\pi i \mathbf{e}_k$  respectively. However, using Proposition 1,

$$\begin{aligned} & \int_{a_k} \lambda(P) d \log F(P; \mathbf{v}) + \int_{b_k} \lambda(P) d \log F(P; \mathbf{v}) \\ & - \int_{a_k \text{ (after } b_k)} \lambda(P) d \log F(P; \mathbf{v}) - \int_{b_k \text{ (after } a_k)} \lambda(P) d \log F(P; \mathbf{v}) \\ & = \int_{a_k} \lambda(P) d \log F(P; \mathbf{v}) + \int_{b_k} \lambda(P) d \log F(P; \mathbf{v}) \\ & \quad - \int_{a_k} \lambda(P) d \log \left[ e^{-\frac{1}{2} B_{kk}} e^{-(j_k(P) - v_k)} F(P; \mathbf{v}) \right] - \int_{b_k} \lambda(P) d \log F(P; \mathbf{v}) \end{aligned}$$

where in the integrals on the second line  $F$  now has the same value as in the corresponding integrals on the first line. Therefore, since by definition of the Abel mapping,

$$dj_k(P) = \omega_k(P),$$

we have

$$J = \frac{1}{2\pi i} \sum_{k=1}^G \int_{a_k}^G \lambda(P) \omega_k(P),$$

a number that depends only on the Riemann surface  $\Gamma$  and not on the vector  $\mathbf{v}$ .

Next, evaluating  $J$  by residues, we note that the differential  $d \log F(P; \mathbf{v})$  has residues equal to 1 at all of the points  $P_1, \dots, P_G$  that are the simple zeros of  $F(P; \mathbf{v})$ . The only other singularity is at the point  $P_\infty$  in  $\tilde{\Gamma}$  corresponding to  $\lambda = \infty$ . Therefore,

$$\begin{aligned} J &= \sum_{k=1}^G \lambda(P_k) + \operatorname{Res}_{P=P_\infty} \lambda(P) d \log F(P; \mathbf{v}) \\ &= \sum_{k=1}^G \mu_k + \operatorname{Res}_{P=P_\infty} \lambda(P) \log F(P; \mathbf{v}), \end{aligned}$$

It remains to calculate the residue at  $P_\infty$ . Since we have a small counterclockwise-oriented contour surrounding  $P_\infty$ , we should introduce a holomorphic local coordinate for  $\Gamma$  near  $\lambda = \infty$  (and hence for  $\tilde{\Gamma}$  near  $P = P_\infty$ ). Since  $\lambda = \infty$  is a square-root branch point for  $\Gamma$ , a holomorphic coordinate that is equal to zero for  $\lambda = \infty$  is

$$k(P) = \frac{1}{(-\lambda(P))^{1/2}}.$$

Then,  $\lambda(P) = -k(P)^{-2}$ , and near infinity, we have

$$y(P) = \frac{k(P)^{2G+1}}{\sqrt{6}} (1 + O(k(P)^2)).$$

To start, we write the components of the vector  $\mathbf{j}(P)$  in terms of  $k$  for  $P$  near  $P_\infty$ :

$$\begin{aligned} j_n(P) &= \int_{P_0}^P \omega_n = \int_{P_0}^{P_\infty} \omega_n + \int_{P_\infty}^P \omega_n \\ &= j_n(P_\infty) + \int_{P_\infty}^P \lambda(P) \frac{H_n(\lambda(P)) d\lambda(P)}{y(P)}. \end{aligned}$$

Now, since

$$H_n(\lambda(P)) = ih_n^{(G-1)} \lambda(P)^{G-1} + O(\lambda(P)^{G-2}) = (-1)^{G-1} ih_n^{(G-1)} k(P)^{2-2G} + O(k(P)^{4-2G})$$

as  $P \rightarrow P_\infty$  (or what's the same,  $k(P) \rightarrow 0$ ), and  $d\lambda(P) = 2k(P)^{-3} dk(P)$ , we get

$$\begin{aligned} j_n(P) &= j_n(P_\infty) + \int_0^k \left[ (-1)^{G-1} 2\sqrt{6} ih_n^{(G-1)} + O(k^2) \right] dk \\ &= j_n(P_\infty) + (-1)^{G-1} 2\sqrt{6} ih_n^{(G-1)} k + O(k^3) \end{aligned}$$

as  $k \rightarrow 0$ . Letting  $\mathbf{g}$  denote the vector

$$\mathbf{g} := \begin{bmatrix} (-1)^{G-1} 2\sqrt{6} ih_1^{(G-1)} \\ \vdots \\ (-1)^{G-1} 2\sqrt{6} ih_G^{(G-1)} \end{bmatrix},$$

we therefore have, by Taylor expansion,

$$\begin{aligned} \log \theta(\mathbf{j}(P) - \mathbf{v}; \mathbf{B}) &= \log \theta(\mathbf{j}(P_\infty) - \mathbf{v} + \mathbf{g}k + O(k^3); \mathbf{B}) \\ &= \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}; \mathbf{B}) + k(\mathbf{g} \cdot \nabla) \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}; \mathbf{B}) \\ &\quad + \frac{k^2}{2} (\mathbf{g} \cdot \nabla)^2 \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}; \mathbf{B}) + O(k^3), \end{aligned}$$

where by  $\nabla$  we mean the gradient of the theta function with respect to the vector  $\mathbf{w}$  variable. Therefore, in terms of  $k$ ,

$$\lambda(P)d \log \theta(\mathbf{j}(P) - \mathbf{v}; \mathbf{B}) = -\frac{1}{k^2} [(\mathbf{g} \cdot \nabla) \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}; \mathbf{B}) + k(\mathbf{g} \cdot \nabla)^2 \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}; \mathbf{B}) + O(k^2)] dk,$$

so picking off the coefficient of  $dk/k$  we get

$$\mathop{Res}_{P=P_\infty} \lambda(P)d \log F(P; \mathbf{v}) = -(\mathbf{g} \cdot \nabla)^2 \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}; \mathbf{B}).$$

Equating the two formulae for  $J$  gives the identity

$$\frac{1}{2\pi i} \sum_{k=1}^G \int_{a_k}^G \lambda(P)\omega_k(P) = \sum_{k=1}^G \mu_k - (\mathbf{g} \cdot \nabla)^2 \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}; \mathbf{B}).$$

Recall that  $\mathbf{v} = \mathbf{v}(x, t)$  is known explicitly. Therefore a formula for  $u(x, t)$  solving KdV is

$$\begin{aligned} u(x, t) &= 2 \sum_{k=1}^G \mu_k(x, t) - \sum_{n=0}^{2G} \lambda_n \\ &= 2(\mathbf{g} \cdot \nabla)^2 \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}(x, t); \mathbf{B}) + \frac{1}{\pi i} \sum_{k=1}^G \int_{a_k}^G \lambda(P)\omega_k(P) - \sum_{n=0}^{2G} \lambda_n. \end{aligned}$$

As a final step, we rewrite the directional derivative  $\mathbf{g} \cdot \nabla$  in terms of a partial derivative with respect to  $x$ . Taking an  $x$ -derivative using the chain rule gives

$$\frac{\partial}{\partial x} \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}(x, t); \mathbf{B}) = \frac{(-1)^{G-1}}{\sqrt{6}} \mathbf{g} \cdot \nabla \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}(x, t); \mathbf{B}),$$

so,

$$u(x, t) = 12 \frac{\partial^2}{\partial x^2} \log \theta(\mathbf{j}(P_\infty) - \mathbf{v}(x, t); \mathbf{B}) + c,$$

where the constant  $c$  is

$$c := \frac{1}{\pi i} \sum_{k=1}^G \int_{a_k}^G \lambda(P)\omega_k(P) - \sum_{n=0}^{2G} \lambda_n.$$

This formula for  $u(x, t)$  is called the *Its-Matveev formula*. Note its similarity to the Kay-Moses formula. Indeed, the Its-Matveev formula has the form

$$u(x, t) = 12 \frac{\partial^2}{\partial x^2} \log \tau(x, t),$$

where

$$\tau(x, t) = e^{\frac{c}{24}x^2} \theta(\mathbf{j}(P_\infty) - \mathbf{v}(x, t); \mathbf{B}).$$

*The character of the Its-Matveev solution formula.* The numbers  $h_k^{(n)}$  are purely real. This means that the motion in the argument of the theta function is linear motion in the purely imaginary directions of  $\mathbb{C}^G$ . This in turn, via the  $2\pi$ -periodicity of the theta function in each of the directions  $i\mathbf{e}_k$  (see Proposition 1), shows that  $u(x, t)$  is a multiperiodic (or quasiperiodic) function of  $x$  and  $t$ . We have already seen in the case  $G = 1$  that  $u(x, t)$  is  $2\pi$ -periodic in a phase variable of the form  $k_1x - \omega_1t$ . For more general  $G$ ,  $u(x, t)$  is a  $2\pi$ -periodic function in each of  $G$  independent phase variables of the form  $k_jx - \omega_jt$  for  $j = 1, \dots, G$ . We say that  $u(x, t)$  is a *multiphase wave* solution of KdV.

One way to get your head around the way such a solution should look is to think again about the  $G = 1$  (elliptic function) case when the elliptic modulus  $m$  is close to  $m = 1$ . In this situation, the solution  $u(x, t)$  is a periodic traveling wave of constant velocity with a very, very large period, and where  $u(x, t)$  is not constant it is very close to a squared hyperbolic secant function. In other words, in the  $G = 1$  case with  $m$  close to  $m = 1$  the solution  $u(x, t)$  is a train of co-propagating, equally-spaced and well-separated  $\text{sech}^2$  solitons. The case of  $m$  close to  $m = 1$  meant that the finite band  $[\lambda_0, \lambda_1]$  was nearly shrunk to a point. For more general  $G$  we may consider the situation in which all finite bands  $[\lambda_0, \lambda_1], [\lambda_2, \lambda_3], \dots, [\lambda_{2G-2}, \lambda_{2G-1}]$  are very short intervals. In this situation, one will have  $G$  trains of well-separated solitons. There are  $G$  different soliton amplitudes and  $G$  different corresponding soliton velocities, so occasionally the solitons of

one train will overtake those of another train. These pairwise overtakings will occur in a manner described approximately by the two-soliton interaction in the Kay-Moses formula with  $N = 2$ , with coincident phase shifts. These overtaking events will occur quasiperiodically in time and space.