(1) (a)Perform the indicated calculation, and give your answer in the simplified form $z = x + iy$:

$$z = \frac{5+i}{-5+i} = \frac{5+i}{-5+i} \cdot \frac{-5-i}{-5-i} = \frac{-(24+10i)}{26} = \frac{-12}{13} + \frac{5}{13}i \ .$$

$$\text{or, from the other exam:} \quad z = \frac{1+7i}{-1+7i} = \frac{24}{25} - \frac{7}{25}i \ .$$

(1b) Write the number $z = 1 + i\sqrt{3}$ in polar form $Re^{i\theta}$, then compute $z^3 = \left(1 + i\sqrt{3}\right)^3$.

$$R = 2, \quad \theta = \frac{\pi}{3}, \quad z = 2e^{\frac{i\pi}{3}} \ , \qquad z^3 = 8e^{-\pi} = -8 \ .$$

$$\text{or, from the other exam:} \quad z = 1 - i\sqrt{3} = 2e^{\frac{-i\pi}{3}}, \quad z^3 = 8e^{-i\pi} = -8 \ .$$

(2) The function $f(x)$ is approximated near $x = 0$ by the third degree Taylor polynomial

$$P_3(x) = 2 - 4x^2 + 2x^3.$$

(a) Give the exact value of $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

$$f(0) = 2, \quad f'(0) = 0, \quad f''(0) = -8, \quad f'''(0) = 12 \ .$$

$$\text{or, from the other exam:} \quad f(0) = 2, \quad f'(0) = 0, \quad f''(0) = 8, \quad f'''(0) = 12 \ .$$

(b) Is $x = 0$ a local maximum, local minimum, or neither? Since the first derivative is zero at $x = 0$, and since the second derivative is negative, $x = 0$ is a local maximum. (On "the other exam", the second derivative was positive, so $x = 0$ was a local minimum.)

(c) Sketch a graph of $f(x)$ **in a vicinity of** $x = 0$.

(3) (a) Find the Taylor polynomial of degree 3 about $a = 8$ for the function $f(x) = x^{1/3}$.

(b) Using your answer to part (a), find an approximation to the cube root of 10, i.e. $10^{1/3}$. *You should compare your answer to the exact value of* $10^{1/3}$ *provided by your calculator.*

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8)^3 \ ,$$

$$P_3(10) = 2 + \frac{1}{6} - \frac{1}{72} + \frac{5}{2592} = \frac{5585}{2592} \approx 2.15471$$

$$10^{1/3} = 2.154434...$$

(4) (a) Do the following series converge or diverge? Explain by making reference to one of the tests for convergence (if it converges) or explain why it diverges.

$$\sum_{n=1}^{\infty} \frac{2n}{\sqrt{2+n^2}}, \qquad \sum_{n=1}^{\infty} \frac{2^n}{3^n + 5}$$

For the first series, the terms in the series do not go to zero as $n \to \infty$, and so the series diverges. For the second series, we have $\frac{2^n}{3^n+5} < \left(\frac{2}{3}\right)^n$, and since the series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a geometric series and converges since $\frac{2}{3} < 1$, the original series converges.

(b) Determine the radius of convergence for the following series. Show your work.

$$\sum_{n=1}^{\infty} \frac{x^n}{3^n \sqrt{n}}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{3^{n+1}} \frac{3^n}{x^n} \frac{\sqrt{n}}{\sqrt{n+1}} \right| = \left| \frac{x}{3} \right| \sqrt{\frac{n}{n+1}} \to \left| \frac{x}{3} \right| \ .$$

So, for convergence, we need $\left|\frac{x}{3}\right| < 1$ which means $|x| < 3$, and so the radius is 3. (For the other exam, the series was $\sum_{n=1}^{\infty} \frac{x^n}{4^n \sqrt{n}}$, and so the radius turns out to be 4.)

(c) Does the above series converge if you plug in $x = -3$? Explain your answer. Plugging in $x = -3$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which is alternating. The terms clearly alternate, and as $n \to \infty$, $\frac{1}{\sqrt{n}} \to 0$. Thus this alternating series converges.

Bonus (3pts). A tennis ball is dropped from a height of 40 feet and bounces. Each bounce is $\frac{1}{2}$ the height of the bounce before. A "superball" has a bounce $\frac{3}{4}$ the height of the bounce before, and is dropped from a height of 30 feet. Which ball travels a greater total vertical distance? For this problem one had to determine the exact amount each ball traveled, and then compare. The tennis ball traveled 120 feet, and the superball traveled 210 feet.

Bonus (2pts). TRUE/FALSE questions.

(a) If a power series $\sum a_k x^k$ conveges at $x = 1$ and at $x = 2$, then it converges at $x = -1$. True

(b) If a power series $\sum a_k x^k$ diverges at $x = c$ then it also diverges at $x = -c$. False