Solutions to selected problems: Homework 1

Math 223 · Section 12 · Fall 2015 Dr. Gilbert

1. Find a vector of length 2 that points in the same direction as $\hat{i} - \hat{j} + 2\hat{k}$

Solution: Let $\vec{v} = \hat{i} - \hat{j} + 2\hat{k}$. First, we can find a unit vector that points in the same direction as $\vec{v}$: We have

$$||\vec{v}|| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}.$$

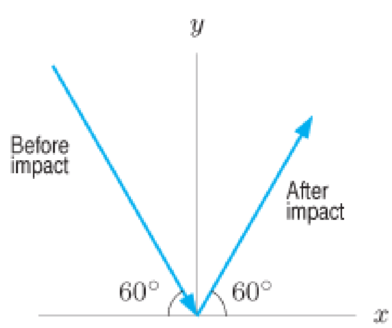
Thus, a unit vector pointing in the direction of $\vec{v}$ is given by

$$\hat{v} = \frac{\vec{v}}{||v||} = \frac{1}{\sqrt{6}}\hat{i} - \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}.$$

To obtain the desired vector, we simply scale $\hat{v}$ by a factor of 2:

$$\vec{w} = 2\hat{v} = \frac{2}{\sqrt{6}}\hat{i} - \frac{2}{\sqrt{6}}\hat{j} + \frac{4}{\sqrt{6}}\hat{k}.$$

2. A particle moving with speed v hits a barrier at an angle of 60° and bounces off at an angle of 60° in the opposite direction with speed reduced by 20%. See the figure below. Find the velocity vector of the object after impact.



Solution: After impact, the speed of the particle is reduced by 20%. Therefore, since the original speed was v, the final speed will be $0.8v$. Using a little bit of trigonometry, and resolving the final velocity vector, $\vec{v}_f$, into components, we get

$$\vec{v}_f = 0.8v\cos(\frac{\pi}{3})\hat{i} + 0.8v\sin(\frac{\pi}{3})\hat{j} = 0.4v\hat{i} + 0.4v\sqrt{3}\hat{j}.$$

3. Given $\vec{v} = 3\hat{\imath} + 4\hat{\jmath}$ and force vector $\vec{F} = -3\hat{\imath} - 5\hat{\jmath}$, find
   (a) The component of $\vec{F}$ parallel to $\vec{v}$.
   (b) The component of $\vec{F}$ perpendicular to $\vec{v}$.
   (c) The work, W, done by force $\vec{F}$ through displacement $\vec{v}$.

   Solution:

   (a) Recall that $\vec{F}_{\text{parallel}} = (\vec{F} \cdot \hat{v})\,\hat{v}$, where $\hat{v}$ is a unit vector pointing in the same direction as $\vec{v}$. So first we find $\hat{v}$. We have

$$||\vec{v}|| = \sqrt{3^2 + 4^2} = 5,$$

   so that

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||} = \frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}.$$

   Finally, we perform the main calculations:

$$\begin{aligned}
\vec{F}_{\text{parallel}} &= (\vec{F} \cdot \hat{v})\,\hat{v} \\
&= (-3\hat{\imath} - 5\hat{\jmath}) \cdot (\frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath})\,\hat{v} \\
&= (-\frac{9}{5} - \frac{20}{5})\,\hat{v} \\
&= -\frac{29}{5}(\frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}) \\
&= -\frac{87}{25}\hat{\imath} - \frac{116}{5}\hat{\jmath}.
\end{aligned}$$

   Thus, we have shown that $\vec{F}_{\text{parallel}} = -\frac{87}{5}\hat{\imath} - \frac{116}{5}\hat{\jmath}$.

   (b) We have $\vec{F}_{\text{perpendicular}} = \vec{F} - \vec{F}_{\text{parallel}}$. So

$$\vec{F}_{\text{perpendicular}} = -3\hat{\imath} - 5\hat{\jmath} - (-\frac{87}{25}\hat{\imath} - \frac{116}{25}\hat{\jmath}) = \frac{12}{25}\hat{\imath} - \frac{9}{25}\hat{\jmath}.$$

   (c) To find the work done by $\vec{F}$ over the displacement $\vec{v}$, we need only calculate

$$W = \vec{F} \cdot \vec{v} = (-3\hat{\imath} - 5\hat{\jmath}) \cdot (3\hat{\imath} + 4\hat{\jmath}) = -9 - 20 = -29.$$

   Note that we have neglected to include units, as the units were not provided to us in the problem.

4. Suppose $\vec{v} \cdot \vec{w} = 8$ and $\vec{v} \times \vec{w} = 12\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and that the angle between $\vec{v}$ and $\vec{w}$ is θ. Find $\tan\theta$ and θ.

Solution: The strategy here is to utilize the geometric definitions of the dot product and cross product. i.e. we know that $\vec{v} \cdot \vec{w} = ||\vec{v}||\,||\vec{w}||\cos\theta$ and $||\vec{v} \times \vec{w}|| = ||\vec{v}||\,||\vec{w}||\sin\theta$. Thus, we have (for any two non-zero vectors $\vec{v}$ and $\vec{w}$)

$$\begin{aligned}\frac{||\vec{v} \times \vec{w}||}{\vec{v} \cdot \vec{w}} &= \frac{||\vec{v}||\,||\vec{w}||\sin\theta}{||\vec{v}||\,||\vec{w}||\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta.\end{aligned}$$

Therefore, for our vectors, we have $||\vec{v} \times \vec{w}|| = \sqrt{12^2 + (-3)^2 + 4^2} = 13$, so

$$\tan\theta = \frac{||\vec{v} \times \vec{w}||}{\vec{v} \cdot \vec{w}} = \frac{13}{8},$$

so we have found that $\tan\theta = \frac{13}{8}$ and therefore $\theta = \arctan\left(\frac{13}{8}\right) \approx 58.39°$.

5. Show that $\left|\left|\vec{a} \times \vec{b}\right|\right|^2 = ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 - (\vec{a} \cdot \vec{b})^2$. We show that the right hand side is equal to the left hand side. We have

$$\begin{aligned}||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 - (\vec{a} \cdot \vec{b})^2 &= ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 - ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 \cos^2\theta \\ &= ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 (1 - \cos^2\theta) \\ &= ||\vec{a}||^2 \left|\left|\vec{b}\right|\right|^2 \sin^2\theta \\ &= \left|\left|\vec{a} \times \vec{b}\right|\right|^2.\end{aligned}$$