Derivative Rules & Higher Derivatives

Sections 3.2 & 3.5
September 23, 2013
Last time we only introduced two derivative rules, the **power** and **sum/difference** rule.

**The Power Rule**

If $f(x) = cx^n$ for any real numbers $c$ and $n$, then

$$f'(x) = cnx^{n-1}.$$  

**Sum/Difference Rule**

Suppose $h(x) = f(x) \pm g(x)$ and both $f'(x)$ and $g'(x)$ exist. Then

$$h'(x) = f'(x) \pm g'(x).$$

Next, we want to introduce the **product** and **quotient** rule.
The Product Rule

Let \( h(x) = f(x)g(x) \) and suppose both \( f'(x) \) and \( g'(x) \) are defined, then

\[
h'(x) = f'(x)g(x) + f(x)g'(x)
\]

Proof:

\[
\frac{d}{dx}(f(x)g(x)) = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x + h)g(x) + f(x + h)g(x) - f(x)g(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x + h)g(x)}{h} + \lim_{h \to 0} \frac{f(x + h)g(x) - f(x)g(x)}{h}
\]

\[
= \lim_{h \to 0} f(x + h) \frac{g(x + h) - g(x)}{h} + \lim_{h \to 0} g(x) \frac{f(x + h) - f(x)}{h}
\]

\[
= f(x) g'(x) + g(x) f'(x).
\]
EXAMPLES

Determine the following:

(1) \( \frac{df}{dx} \) for \( f(x) = (3x^2 + 2)(2x - 1) \)

\[
\frac{df}{dx} = (6x)(2x - 1) + (3x^2 + 2)(2)
\]

(2) \( \frac{d}{dx} \left[ x^{-2}(4 + 3x^{-3}) \right] \)

\[
\frac{d}{dx} \left[ x^{-2}(4 + 3x^{-3}) \right] = (-2x^{-3})(4 + 3x^{-3}) + (x^{-2})(-9x^{-4})
\]

(3) \( D_x[(x^3 + 7x - 1)(5x + 2)] \)

\[
D_x[(x^3 + 7x - 1)(5x + 2)] = (3x^2 + 7)(5x + 2) + (x^3 + 7x - 1)(5)
\]
The Quotient Rule

Let
\[ h(x) = \frac{f(x)}{g(x)} \quad \text{(so } g(x) \neq 0) \]

and suppose and both \( f'(x) \) and \( g'(x) \) are defined. Then
\[ h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \]
Proof of the Quotient Rule

Proof:

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{g(x + h) - g(x)} = \lim_{h \to 0} \frac{g(x)f(x + h) - g(x)f(x)}{g(x)g(x + h) - g(x)g(x)}
\]

\[
= \lim_{h \to 0} \frac{g(x)f(x + h) - g(x + h)f(x)}{hg(x)g(x + h)}
\]

\[
= \lim_{h \to 0} \frac{g(x)f(x + h) - f(x)g(x) + f(x)g(x) - g(x + h)f(x)}{hg(x)g(x + h)}
\]

\[
= \lim_{h \to 0} \frac{g(x)f(x + h) - f(x)g(x) + f(x)g(x) - g(x + h)f(x)}{hg(x)g(x + h)}
\]

\[
= \lim_{h \to 0} \frac{g(x)f(x + h) - f(x)g(x)}{h} \cdot \frac{g(x) - g(x + h)}{g(x)g(x + h)}
\]

\[
= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)g(x + h)}
\]

\[
= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.
\]
EXAMPLES

Determine the following:

1. \( \frac{df}{dx} \) for \( f(x) = \frac{6x + 1}{3x + 10} \)

\[
\frac{df}{dx} = \frac{(6)(3x + 10) - (6x + 1)(3)}{(3x + 10)^2}
\]

2. \( \frac{d}{dx} \left[ \frac{x^2 + x}{x - 1} \right] \)

\[
\frac{d}{dx} \left[ \frac{x^2 + x}{x - 1} \right] = \frac{(2x + 1)(x - 1) - (x^2 + x)(1)}{(x - 1)^2}
\]

3. \( D_t \left[ \frac{\sqrt{t}}{t^2 - 1} \right] \)

\[
D_t \left[ \frac{\sqrt{t}}{t^2 - 1} \right] = \frac{(\frac{1}{2}t^{-1/2})(t^2 - 1) - (\sqrt{t})(2t)}{(t^2 - 1)^2}
\]
Practice Problems
(1) Determine
\[
\frac{d}{dy} \left[ (y^{-1} + y^{-2})(2y^{-3} - 5y^{-4}) + \frac{(3y^2 + 1)(2y - 1)}{5y + 4} \right]
\]

(2) If \( g(2) = 3, h(2) = 5, g'(2) = 7, \) and \( h'(2) = 14, \) find \( f'(2) \) for
\[
f(x) = g(x)h(x).
\]

(3) Find the equation of the line tangent to the graph of
\[
f(x) = \frac{x}{x - 2}
\]
at the point \((3, 3)\).
The Chain Rule
In math there are many ways to combine two functions to get another function.

The four most obvious are:

- **addition:** $f+g$
- **subtraction:** $f - g$
- **multiplication:** $fg$
- **division:** $f/g$

We know how to find the derivative of each of these combinations.

Next we’re going to introduce a fifth way of combining functions and a derivative rule for it.
Composition of Functions

Let $f$ and $g$ be functions. The **composition** of $f$ with $g$, denoted $f \circ g$, is given by

$$(f \circ g)(x) = f(g(x))$$

For example, suppose $f(x) = \frac{x}{8} + 7$ and $g(x) = 6x - 1$.

Determine $(f \circ g)$ and $(g \circ f)$:

$$(f \circ g)(x) = f(g(x)) = f(6x - 1) = \frac{6x - 1}{8} + 7 = \frac{3}{4}x + \frac{55}{8}$$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{8} + 7\right) = 6\left(\frac{x}{8} + 7\right) - 1 = \frac{3}{4}x + 41$$
We’ll actually be more interested in going in the other direction: Given a function $h$ we want to express it as the composition of two functions $f$ and $g$. There are usually **MANY** ways to do this.

Write each of the following as the composition of two function $f$ and $g$:

(1) $h(x) = \frac{1}{x^2}$

$h(x) = (f \circ g)(x)$ for $f(x) = \frac{1}{x}$ and $g(x) = x^2$

(2) $h(x) = (x^3 + 2)^2 + 4(x^3 + 2) + 5$

$h(x) = (f \circ g)(x)$ for $f(x) = x^2 + 4x + 5$ and $g(x) = x^3 + 2$
The Chain Rule

Let \( h(x) = (f \circ g)(x) \) for two functions \( f \) and \( g \). Then

\[
h'(x) = f'(g(x)) \cdot g'(x)
\]

(1) Determine the derivative of \( h(x) = (x + 1)^4 \).

**First:** Express \( h \) as the composition of two functions \( f \) and \( g \):

\[
f(x) = x^4 \quad \quad g(x) = x + 1
\]

\[
(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^4 = h(x)
\]

**Second:** Apply the chain rule. Since

\[
f'(x) = 4x^3 \quad \quad g'(x) = 1
\]

The chain rule tells us that

\[
h'(x) = f'(g(x)) \cdot g'(x) = 4(x + 1)^3 \cdot 1 = 4(x + 1)^3.
\]
(2) Determine the derivative of \( r(t) = t(2t^2 + 1)^4 \).

**First:** We start with the product rule

\[
r'(t) = (1)(2t^2 + 1)^4 + (t) \left( \frac{d}{dt}[(2t^2 + 1)^4] \right)
\]

**Second:** Express \((2t^2 + 1)^4\) as the composition of two functions:

\[
f(t) = t^4 \quad g(t) = 2t^2 + 1
\]

\[
(f \circ g)(t) = f(g(t)) = f(2t^2 + 1) = (2t^2 + 1)^4
\]

**Third:** Apply the chain rule. Since

\[
f'(t) = 4t^3 \quad g'(t) = 4t
\]

\[
\frac{d}{dt}[(2t^2 + 1)^4] = f'(g(t)) \cdot g'(t) = 4(2t^2 + 1)^3 \cdot 4t = 16t(2t^2 + 1)^3
\]

So,

\[
r'(t) = (2t^2 + 1)^4 + 16t^2(2t^2 + 1)^3
\]
(3) Determine the derivative of $h(t) = \frac{t^2 + 1}{t - 1}$.

**First:** We rewrite $h(t) = (t^2 + 1)(t - 1)^{-1}$ and use the product rule

$$h'(t) = (2t)(t - 1)^{-1} + (t^2 + 1) \left( \frac{d}{dt}[(t - 1)^{-1}] \right)$$

**Second:** We express $(t - 1)^{-1}$ as the composition of two functions:

$$f(t) = t^{-1} \quad g(t) = t - 1$$

$$(f \circ g)(t) = f(g(t)) = f(t - 1) = (t - 1)^{-1}$$

**Third:** Apply the chain rule. Since

$$f'(t) = -t^{-2} \quad g'(t) = 1$$

$$\frac{d}{dt}[(t - 1)^{-1}] = f'(g(t)) \cdot g'(t) = -(t - 1)^{-2} \cdot 1 = -(t - 1)^{-2}$$

So

$$(2t)(t - 1)^{-1} - (t^2 + 1)(t - 1)^{-2}$$
You may have already realized the following consequence of the chain rule:

**Generalized Power Rule**

Let $c$ and $n$ be real numbers, then

$$
\frac{d}{dx}(f(x)^n) = n \cdot f(x)^{n-1} \cdot f'(x)
$$

**Proof:** First we note that if $h(x) = x^n$

$$(h \circ f)(x) = h(f(x)) = f(x)^n$$

So, by the chain rule we have

$$
\frac{d}{dx}(f(x)^n) = \frac{d}{dx}((h \circ f)(x)) = n \cdot f(x)^{n-1} \cdot f'(x).
$$
Practice Problems
COMPOSITION

Determine the derivatives of the following functions using the chain rule:

(1) \( f(x) = (2x^3 + 9x)^5 \)
(2) \( g(y) = 4y^2(y^2 + 1)^{5/4} \)
(3) \( h(t) = \frac{1}{(x^3 + 1)^3} \)
(4) \( f(x) = \frac{x^2 + 4x}{(3x^2 + 1)^4} \)

Use the following table to determine the values to the right.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>-6</td>
<td>-7</td>
<td>-8</td>
<td>-9</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>( g'(x) )</td>
<td>2/7</td>
<td>3/7</td>
<td>4/7</td>
<td>5/7</td>
</tr>
</tbody>
</table>

(i) \( D_x(g \circ f) \) at \( x = 1 \)
(ii) \( D_x(f \circ g) \) at \( x = 1 \)
(iii) \( D_x(g \circ f) \) at \( x = 2 \)
(iv) \( D_x(f \circ g) \) at \( x = 2 \)
Next we want to consider the **higher derivatives** of a function. These will just be **derivatives of derivatives**.

For example, let \( f(x) = x^5 - 3x^2 + 12x - 1 \). Then

\[
f'(x) = 5x^4 - 6x + 12 \quad \text{first derivative of } f
\]

Taking the derivative of \( f'(x) \) we get

\[
f''(x) = 20x^3 - 6 \quad \text{second derivative of } f
\]

Continuing on...

\[
f'''(x) = 60x^2 \quad \text{third derivative of } f
\]

\[
f^{(4)}(x) = 120x \quad \text{fourth derivative of } f
\]

\[
f^{(5)}(x) = 120 \quad \text{fifth derivative of } f
\]

\[
f^{(6)}(x) = 0 \quad \text{sixth derivative of } f
\]
We denote the 1\textsuperscript{st}, 2\textsuperscript{nd} and 3\textsuperscript{rd} derivatives of a function $f$ by:

$$f'(x), \quad \frac{d}{dx}(f(x)), \quad \frac{df}{dx}, \quad D_x[f(x)]$$

first derivative of $f$

$$f''(x), \quad \frac{d^2}{dx^2}(f(x)), \quad \frac{d^2f}{dx^2}, \quad D_x^2[f(x)]$$

second derivative of $f$

$$f'''(x), \quad \frac{d^3}{dx^3}(f(x)), \quad \frac{d^3f}{dx^3}, \quad D_x^3[f(x)]$$

third derivative of $f$

For $n > 3$ we denote the $n\textsuperscript{th}$ derivative by

$$f^{(n)}(x), \quad \frac{d^n}{dx^n}(f(x)), \quad \frac{d^n f}{dx^n}, \quad D_x^n[f(x)]$$

$n\textsuperscript{th}$ derivative of $f$
The information given by these higher derivatives depends on the situation.

However, there is one particular situation you should be familiar with:

\[ P(t) \]  The **position** of an object at time \( t \)

\[ P'(t) \]  The **velocity** of the object at time \( t \)

\[ P''(t) \]  The **acceleration** of the object at time \( t \)
A baseball is thrown straight up from an initial height of 32 feet. It’s height $t$ seconds after it is thrown is given by:

$$H(t) = -16t^2 + 48t + 32.$$  

(1) Determine the functions giving the velocity and acceleration of the baseball at time $t$.

The **velocity** is given by:

$$H'(t) = -32t + 48$$

The **acceleration** is given by:

$$H''(t) = -32$$
A baseball is thrown straight up from an initial height of 32 feet. It’s height $t$ seconds after it is thrown is given by:

$$H(t) = -16t^2 + 48t + 32.$$  

(2) How high does the baseball go?

The rock reaches its maximum height when its velocity is zero (its height would be either increasing or decreasing if its velocity wasn’t 0).

So, we need to determine when velocity is 0, i.e when $H'(t) = 0$.

$$H'(t) = -32t + 48 = 0$$

$$\Rightarrow t = \frac{-48}{-32} = \frac{3}{2} \text{ seconds}$$
A baseball is thrown straight up from an initial height of 32 feet. Its height $t$ seconds after it is thrown is given by:

$$H(t) = -16t^2 + 48t + 32.$$  

(3) What is the velocity of the baseball when it is 52 feet above the ground on its way up?

First we need to determine when the baseball is 52 feet above the ground on its way up, i.e. when $H(t) = 42$.

$$-16t^2 + 48t + 32 = 52 \implies -16t^2 + 48t - 20 = 0$$

$$\implies t = \frac{-48 \pm \sqrt{48^2 - 4(-16)(-20)}}{-32} = 0.5, 2.5$$

So, the velocity of the baseball when it is 52 feet above the ground on its way up is:

$$H'(0.5) = -16 + 48 = 32 \text{ ft./sec.}$$
A baseball is thrown straight up from an initial height of 32 feet. Its height \( t \) seconds after it is thrown is given by:

\[
H(t) = -16t^2 + 48t + 32.
\]

(4) What is the acceleration of the baseball when it is 10 feet above the ground on its way down?

First we need to determine when the baseball is 52 feet above the ground on its way down.

However, from the last problem we know this is when \( t = 2.5 \).

So, the acceleration is:

\[
H''(2.5) = -32 \text{ ft./sec}^2
\]