Implicit Functions

There are two ways of defining a function, explicitly and implicitly. Up to this point, most of the functions that we’ve dealt with have been defined explicitly.

For example,

\[ y = x^2 - 1 \]

defines the \( y \) explicitly as a function of \( x \), while

\[ 1 = x^2 - y \]

defines the same function implicitly.

Intuitively, \( y \) is defined explicitly as a function of \( x \) if

\[ y = f(x) \]

for some function of \( x \). If this is not the case we say that \( y \) is defined implicitly as a function of \( x \).

Let’s look at a few more examples to make sure we get the idea:

- **y defined explicitly:**
  \[ y = \frac{3}{x^2} \]

- **y defined implicitly:**
  \[ yx^2 = 3 \]

  \[ y = 5 \cos^2(x) \]

  \[ \cos^2(x) = \frac{y}{5} \]

  \[ y = \frac{1}{x} \]

  \[ x = \frac{1}{y} \]

There are many ways to express a function implicitly, but there is at most one way to express it explicitly. In fact, it is often difficult or impossible to express an implicitly defined function explicitly. For example, if

\[ y^2 + x^2 = 1 \]

we cannot be write \( y \) as an explicit function of \( x \), since solving for \( y \) gives you

\[ y = \pm \sqrt{1 - x}. \]

In this section we will learn how to determine the derivative of implicitly defined functions, even those that cannot be expressed explicitly.
Implicit Differentiation

Let’s consider the implicitly defined function from earlier,

\[ 1 = x^2 - y. \]

Suppose we want to determine the derivative of this function, that is, we want to determine \( \frac{dy}{dx} \) or \( y' \). Well, one way to do it would be to solve for \( y \) and take the derivative:

\[
1 = x^2 - y \\
\Rightarrow y = x^2 - 1 \\
\Rightarrow y' = 2x.
\]

However, we can determine the derivative from its implicit definition if we just think of \( y \) as a function of \( x \) and take the derivative of both sides with respect to \( x \):

\[
1 = x^2 - y \\
\Rightarrow \frac{d}{dx}(1) = \frac{d}{dx}(x^2 - y) \\
\Rightarrow 0 = 2x - y' \\
\Rightarrow y' = 2x.
\]

We get the same answer, but the second method doesn’t require us to express \( y \) explicitly as a function of \( x \). Finding a derivative in this way is called **implicit differentiation**.

Let’s look at a few more examples:

(1) \( yx^2 = 3 \).

\[
\frac{d}{dx}(yx^2) = \frac{d}{dx}(3) \quad \Rightarrow \quad (y')(x^2) + (y)(2x) = 0 \quad \Rightarrow \quad y' = -\frac{2y}{x}
\]

Notice that we still have a \( y \) on the right hand side. That’s perfectly okay. Actually, in this case we could solve for \( y \) and substitute what we get (\( y = 3/x^2 \)) for \( y \). However, that kind of defeats the purpose of using implicit differentiation, and later we’ll see examples where we cannot get rid of the \( y \) in the definition of \( y' \).
(2) \( \cos^2(x) = y/5 \).
\[
\frac{d}{dx} (\cos^2(x)) = \frac{d}{dx} \left( \frac{y}{5} \right) \implies -2 \cos(x) \sin(x) = \frac{1}{5} y'
\]
\[
\implies y' = -10 \cos(x) \sin(x)
\]

(2) \( x = 1/y \).
\[
\frac{d}{dx} (x) = \frac{d}{dx} (y^{-1}) \implies 1 = -y^{-2} y'
\]
\[
\implies y' = -y^2
\]

Note that we had to use the chain rule with respect to \( y \) for this one.

Be for we look at more examples, let’s write down some general steps for determining the derivative of a function \( y \) using implicit differentiation.

**Implicit Differentiation:**

*Step 1:* Take the derivative of both sides with respect to \( x \) (or whatever the variable is), treating \( y \) as a function of \( x \).

*Step 2:* Move all the terms involving \( y' \) to one side of the equation, and everything else to the other.

*Step 3:* Solve for \( y' \).

Now, let’s try this approach with some implicitly functions \( y \) that would be either very difficult or impossible to express explicitly.

(1) \( x^2 + y^2 = 1 \)
\[
\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (1) \implies 2x + 2yy' = 0 \implies y' = -\frac{x}{y}.
\]

(2) \( \theta = \sin(y) \)
\[
\frac{d}{d\theta} (\theta) = \frac{d}{d\theta} (\sin(y)) \implies 1 = \cos(y) y' \]
\[
\implies y' = \frac{1}{\cos(y)}
\]
(3) \( y^2 = e^y + 2w \)

\[
\frac{d}{dw}(y^2) = \frac{d}{dw}(e^y + 2w) \implies 2yy' = e^y y' + 2
\]

\[
\implies 2yy' - e^y y' = 2 \implies y'(2y - e^y) = 2 \implies y' = \frac{2}{2y - e^y}
\]

We can also use implicit differentiation to determine higher derivatives.

**Examples:**

(1) \( x^2 + y^2 = 1 \)

From above we know that

\[
y' = -\frac{x}{y}.
\]

To determine \( y'' \) (or equivalently \( d^2y/dx^2 \)), we take the derivative of both sides of the above equation with respect to \( x \):

\[
\frac{d}{dx}(y') = \frac{d}{dx}\left(-\frac{x}{y}\right) \implies y'' = \frac{(-1)(y) - (-x)(y')}{y^2}
\]

\[
\implies y'' = \frac{xy' - y}{y^2}
\]

We have \( y'' \), but it is define in terms of \( x, y, \) and \( y' \). While we’re perfectly okay with defining \( y'' \) in terms of \( x \) and \( y \), we’d like to get rid of any \( y' \) terms. To do so, we simply use our definition of \( y' \) which is given in terms of \( x \) and \( y \),

\[
y'' = \frac{xy' - y}{y^2} \implies y'' = \frac{x\left(-\frac{x}{y}\right) - y}{y^2} \implies y'' = \frac{-x^2 - y^2}{y^3}
\]

(2) \( \theta = \sin(y) \)

From above we know that

\[
y' = \frac{1}{\cos(y)}.
\]

So,

\[
\frac{d}{d\theta}(y') = \frac{d}{d\theta}\left(\frac{1}{\cos(y)}\right) \implies y'' = \frac{(0)(\cos(y)) - (1)(-\sin(y)y')}{\cos^2(y)}
\]
\[ y'' = \frac{\sin(y)y'}{\cos^2(y)} \]

Substituting our definition of \( y' \), we get

\[ y'' = \frac{\sin(y)}{\cos^3(y)} \]

**Practice Problems**

Determine \( y' \) (or equivalently \( dy/dx \), \( dy/dt \), \( dy/d\theta \), \( dy/dw \), depending on the independent variable) for the following functions:

1. \((3ty + 7)^2 = 6y\)
2. \(\theta y = \cot(\theta y)\)
3. \(e^{w^2}y = 2w + 2y\)

Determine \( y'' \) (or equivalently \( d^2y/dx^2 \)) for the following functions:

4. \(x^2 + y^2 = 1\)
5. \(y^2 = e^{x^2} + 2x\)

**Tangent Lines**

The curves (graphs) that are defined by implicit functions can be much more interesting than those defined by explicit functions.

**Example**: For example, the implicitly defined function

\[ y^2 = x^3 - 3x - 1 \]

is called an **elliptic curve** and graph looks like:
Elliptic curves are a major area of current mathematical research and play a fundamental role in modern cryptography. However, we would just like to be able to determine the equation of a line tangent to this curve at a given point, which we can do using implicit differentiation.

Suppose we want to determine the equation of the line tangent to the curve at the point $(2, 1)$, i.e. the equation of the red line below.
To do so, we first determine \( y' \):
\[
\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 - 3x - 1) \implies 2y'y = 3x^2 - 3 \implies y' = \frac{3x^2 - 3}{2y}.
\]

The slope our of tangent line will be \( y' \) evaluated at the point \((2, 1)\) (notice that we need both an \( x \) and \( y \) value unlike determine the slope of tangent line for an explicitly defined function).

Slope of the tangent line \( = \frac{3(2)^2 - 3}{2(1)} = \frac{9}{2} \).

Next, we simply plug all of this information into the point-slope form for a line, and the equation of our tangent line is:
\[
y = \frac{9}{2}(x - 2) + 1.
\]

Let’s write down the steps we just went through:

**Finding the Tangent Line at a point using Implicit Differentiation:**

*Step 1:* Find \( y' \).

*Step 2:* Plug the \( x \) and \( y \) values of your point into \( y' \) to get the slope of your tangent line.

*Step 3:* Plug the slope and the point into the point-slope form of a line and you’re done!

**Example:** Let’s look at another example. The curve defined by
\[
2(x^2 + y^2)^2 = 25xy^2
\]

is called a **double folium**.
Let's determine the equation of the line tangent to this curve at the point \((2, -1)\).

First, we determine \(y'\):

\[
\frac{d}{dx}(2(x^2 + y^2)^2) = \frac{d}{dx}(25xy^2)
\]

\[
\Rightarrow 4(x^2 + y^2)(2x + 2yy') = (25)(y^2) + (25x)(2yy')
\]

\[
\Rightarrow 4(x^2 + y^2)(2x) + 4(x^2 + y^2)(2yy') = (25)(y^2) + (50xy)y'
\]

\[
\Rightarrow 4(x^2 + y^2)(2yy') - (50xy)y' = (25)(y^2) - 4(x^2 + y^2)(2x)
\]

\[
\Rightarrow y'(8x^2y + 8y^3 - 50xy) = 25y^2 - 8x^3 - 8xy^2
\]

\[
\Rightarrow y' = \frac{25y^2 - 8x^3 - 8xy^2}{8x^2y + 8y^3 - 50xy}
\]

So the slope of the tangent line at \((2, -1)\) is

\[
\frac{25(-1)^2 - 8(2)^3 - 8(2)(-1)^2}{8(2)^2(-1) + 8(-1)^3 - 50(2)(-1)} = \frac{-55}{60} = -\frac{11}{12}
\]

Finally, we plug our slope \(-\frac{11}{12}\) and our point \((2, -1)\) into the point slope form for a line:
The double folium with the tangent line at \((2, -1)\) look like:

![Graph of the double folium with tangent line](image)

**Normal Lines**

Along with the tangent line to the graph at a point, another important line is the **normal line** to the graph at a point.

**Definition 1.** The **normal line to a graph at the point** \((a, b)\) **is the line perpendicular to the tangent line of the graph at the point.**

The equation of this line is given by

\[
y = -\frac{1}{m}(x - a) + b
\]

where \(m\) is the slope of the tangent line at the point \((a, b)\).
So what do these normal lines look like? Well, let’s consider the previous two examples:

**Example:** For \( y^2 = x^3 - 3x - 1 \) we have

Here the *red line* denotes the **tangent line** at \((2, 1)\) and the *green line* denotes the **normal line** at \((2, 1)\).

Since we know the equation of the tangent line at \((2, 1)\) (specifically it’s *slope*), the equation of the normal line at \((2, 1)\) is easy to determine:

First, recall that the slope of the tangent line at \((2, 1)\) is \(9/2\). This means that the normal line at \((2, 1)\) is the line passing through the point \((2, 1)\) with slope \(-2/9\) (the negative reciprocal of the slope of the tangent line). Plugging this into the point-slope form for a line, the equation of the normal line at \((2, 1)\) is:

\[
y = -\frac{2}{9}(x - 2) + 1.
\]
Example: For $2(x^2 + y^2)^2 = 25xy^2$ we have

where once again the red line denotes the tangent line at $(2, 1)$ and the green line denotes the normal line at $(2, 1)$.

Since the equation of the tangent line is:

$$y = -\frac{11}{12}(x - 2) - 1$$

the equation of the normal line is:

$$y = \frac{12}{11}(x - 2) - 1.$$
Practice Problems

(1) Consider the implicitly defined function,
\[ x^2 + xy - y^2 = 1 \]

(i) Which of the following point(s) lie on the curve defined by the function above?
\[ (2, 3) \quad (1, 1) \quad (-1, 2) \quad (1, 0) \]

(ii) Determine the tangent line for each of the above points that lie on the curve.

(2) Consider the implicitly defined function,
\[ y^4 = y^2 - x^2 \]

(i) Which of the following points lie on the curve defined by the function above?
\[ (1, 1) \quad (1/2, \sqrt{3}/4) \quad (\sqrt{3}/4, 1/2) \quad (\sqrt{3}, 1/2) \]

(ii) Determine the tangent line for each of the above points that lie on the curve.