Increasing & Decreasing Functions and Extrema

Section 4.1
October 24, 2013
Limits - Cornerstone of calculus

The Derivative - Gives the rate of change of a function

Derivative Rules - Rules so we don’t have to use the limit definition

Applications of Derivatives - Optimization, finding extreme values
Increasing and Decreasing Functions

\[ f(x) = x^3 - x^2 - 2x + 3.5 \]
INCREASING AND DECREASING FUNCTIONS

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Increasing and Decreasing Functions

$$f(x) = x^3 - x^2 - 2x + 3.5$$

Walking uphill ⇒ function is increasing

Walking downhill ⇒ function is decreasing
**Increasing and Decreasing Functions**

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Walking uphill \( \Rightarrow \) function is **increasing**

Walking downhill \( \Rightarrow \) function is **decreasing**
**INCREASING AND DECREASING FUNCTIONS**

\[ f(x) = x^3 - x^2 - 2x + 3.5 \]

Walking uphill \(\Rightarrow\) function is **increasing**

Walking downhill \(\Rightarrow\) function is **decreasing**

Peak or valley \(\Rightarrow\) Neither increasing nor decreasing
Increasing and Decreasing Functions

Below is the graph of the function $f(x)$.

increasing: $(-3, 0) \cup (2, \infty)$          decreasing: $(-\infty, -3) \cup (0, 2)$
**Definition**

Let $f$ be a function defined on some open interval $(a, b)$.

We say that $f$ is **increasing** on $(a, b)$ if

$$f(x_1) < f(x_2)$$

for all $x_1$ and $x_2$ in $(a, b)$ satisfying $x_1 < x_2$.

$f$ is **decreasing** on $(a, b)$ if

$$f(x_1) > f(x_2)$$

for all $x_1$ and $x_2$ in $(a, b)$ satisfying $x_1 < x_2$. 
INCREASING AND DECREASING FUNCTIONS

Consider the previous graph:

```
-1.5  -1   -0.5  0   0.5  1   1.5  2
-1    1    2    3    4    5
increasing decreasing increasing
```
Increasing and Decreasing Functions

Consider the previous graph:
INCREASING AND DECREASING FUNCTIONS

Consider the previous graph:

positive slope \((f'(x) > 0)\) \(\Rightarrow\) \(f\) is increasing.
Increasing and Decreasing Functions

Consider the previous graph:

\[ f'(x) > 0 \implies f \text{ is increasing}. \]

positive slope \( f'(x) > 0 \) \( \implies \) \( f \) is increasing.
Increasing and Decreasing Functions

Consider the previous graph:

positive slope \((f'(x) > 0)\) \(\Rightarrow\) \(f\) is increasing.

slope of zero \((f'(x) = 0)\) \(\Rightarrow\) \(f\) is neither inc. nor dec.
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Consider the previous graph:

**The Derivative Test**

- **Positive slope** ($f'(x) > 0$) $\Rightarrow$ $f$ is increasing.
- **Slope of zero** ($f'(x) = 0$) $\Rightarrow$ $f$ is neither inc. nor dec..
- **Negative slope** ($f'(x) < 0$) $\Rightarrow$ $f$ is decreasing.
Increasing and Decreasing Functions

Consider the previous graph:

positive slope \((f'(x) > 0)\) \(\Rightarrow\) \(f\) is increasing.

slope of zero \((f'(x) = 0)\) \(\Rightarrow\) \(f\) is neither inc. nor dec.

negative slope \((f'(x) < 0)\) \(\Rightarrow\) \(f\) is decreasing.
Increasing and Decreasing Functions

Consider the previous graph:

- Increasing: When the slope is positive ($f'(x) > 0$) $\Rightarrow f$ is increasing.
- Decreasing: When the slope is negative ($f'(x) < 0$) $\Rightarrow f$ is decreasing.
- Slope of zero: When the slope is zero ($f'(x) = 0$) $\Rightarrow f$ is neither increasing nor decreasing.
INCREASING AND DECREASING FUNCTIONS

Increasing and Decreasing via Derivatives

Suppose \( f(x) \) is a function whose derivative exists at every point in some interval \((a, b)\).

- if \( f'(x) > 0 \) for all \( x \) in \((a, b)\), the function is increasing on \((a, b)\).
- if \( f'(x) < 0 \) for all \( x \) in \((a, b)\), the function is decreasing on \((a, b)\).
- if \( f'(x) = 0 \) for all \( x \) in \((a, b)\), the function is constant on \((a, b)\).
How can we determine intervals where a function is increasing and decreasing from the equation of the function?

If $f'(x)$ goes from positive to negative (or vice versa) at the point $x = c$ in the domain of $f$, then one of two things must be true:

(1) $f'(c) = 0$, or

(2) $f'(x)$ does not exist at $x = c$

We call these $x$-values critical points.
Increasing and Decreasing Functions

Let $f(x) = 2.3 + 3x - x^2$. Determine where the $f$ is increasing and decreasing.

**Step 1: Find all critical points:**

First find $f'(x)$: $f'(x) = 3 - 2x$

Next, determine when $f'(x) = 0$:

$3 - 2x = 0 \Rightarrow 3 = 2x \Rightarrow x = \frac{3}{2}$

Next, determine when $f'(x)$ is undefined:

$f'(x)$ a polynomial $\Rightarrow f'(x)$ defined for all real numbers

Since $x = \frac{3}{2}$ is in the domain of $f$: **Critical point: $x = \frac{3}{2}$**
Let $f(x) = 2.3 + 3x - x^2$. Determine where the $f$ is increasing and decreasing.

**Step 2: Test intervals:**

Our **one** critical number divides the domain of $f$ (all real numbers) into **two intervals**.

- $f$ is **increasing** on $(-\infty, 3/2)$
  - $f'(-1) = 5 > 0$

- $f$ is **decreasing** on $(3/2, \infty)$
  - $f'(2) = -1 < 0$
Practice Problems
Determine the open intervals where the following functions are increasing and decreasing

(1) $g(t) = t^{2/3}$

(2) $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 5$

(3) $h(y) = -3y + 6$
Relative Extrema
Motivating Example

The number of people \( P(t) \) (in hundreds) infected after \( t \) weeks after an epidemic begins is approximated by

\[
P(t) = \frac{10 \ln(t + 1)}{t + 1}.
\]

When will the number of people infected start to decline?

\[
P'(t) = 10 \left( \frac{1}{t+1} \right) (t + 1) - 10 \ln(t + 1)(1) \]
\[
= \frac{10 - 10 \ln(t + 1)}{(t + 1)^2}
\]
Motivating Example

\[ P'(t) = \frac{10 - 10 \ln(t + 1)}{(t + 1)^2} \]

For which \( t \) is \( P'(t) = 0? \)

\[
\frac{10 - 10 \ln(t + 1)}{(t + 1)^2} = 0 \iff 10 - 10 \ln(t + 1) = 0
\]

\[
\implies 10 = 10 \ln(t + 1)
\]

\[
\implies 1 = \ln(t + 1)
\]

\[
\implies e = t + 1
\]

\[
\implies t = e - 1 \quad (\text{This is in the domain of } P, \text{ which is } (-1, \infty))
\]

So we have one **critical number** (so far) when \( t = e - 1 \).
For which $t$ is $P'(t)$ undefined?

There are two things that could cause $P'(t)$ to be undefined:

(1) if $t + 1 < 0$, since the domain of $\ln(x)$ is $(0, \infty)$.

(2) if $(t + 1)^2 = 0$, since we would be dividing by zero.

$t + 1 < 0 \implies t < -1$

$(t + 1)^2 = 0 \implies t = -1$

These values of $t$ are not in the domain of $P(t)$. 

$$P'(t) = \frac{10 - 10 \ln(t + 1)}{(t + 1)^2}$$
Increasing and Decreasing Functions

\[ P'(t) = \frac{10 - 10 \ln(t + 1)}{(t + 1)^2} \]

So, \( P(t) \) has one critical number at \( t = e - 1 \approx 1.71828 \).

This divides the domain of \( P \) into two pieces:

\begin{align*}
\text{\textbf{f is increasing on}} & \quad \text{\textbf{f is decreasing on}} \\
(0, e - 1) & \quad (e - 1, \infty)
\end{align*}

\[ f'(0) = 10 > 0 \quad f'(2) \approx -1.11 < 0 \]

The # of infected people will begin to decline when \( t = e - 1 \).
Let’s take a look at the graph of $P(t)$:

What’s happening at the critical point?
Let’s take a look at the graph of $P(t)$:

What’s happening at the critical point?
Let’s take a look at the graph of $P(t)$:

What’s happening at the critical point? A maximum value.
Relative (Local) Extremum

Let $f$ be a function and let $c$ be a number in its domain.

$f(c)$ is a **relative (local) maximum** if there exists an open interval $(a, b)$ containing $c$ such that

$$f(c) \geq f(x) \text{ for all } x \text{ in } (a, b)$$

$f(c)$ is a **relative (local) minimum** if

$$f(c) \leq f(x) \text{ for all } x \text{ in } (a, b)$$

If $f(c)$ is a local max/min we say it’s a **relative (local) extremum**.
Determine the local extrema for the function graphed below:

**local max:**
3 at $x = 0$

**local min:**
-2 at $x = -2$,
0 at $x = 1$
Find all Relative extrema and where they occur for each of the following functions:

**local max:**
- 0 at $x = -3$,
- 2 at $x = 1$

**local min:**
- $-1$ at $x = 0$,
- $0$ at $x = 2$
What is the relationship between critical points and relative extrema?

Fact

\[ f(c) \text{ is a relative extremum of } f \quad \Rightarrow \quad x = c \text{ is a critical number or endpoint of the domain.} \]
Consider $f(x) = x^3$:

$$f'(x) = 3x^2$$

Critical point at $x = 0$.

$f$ does NOT have a relative max or min at $x = 0$. 

**WARNING!**

$x = c$ is a **critical number** $\implies f(c)$ is a **relative extremum**
How do we determine if a critical point or the endpoint of the domain of a function $f$ gives a relative extremum?

We use the "**First Derivative Test**".

Let $f(x) = x^2 - 10x + 33$. Determine where $f$ is inc./dec. and find any relative extremum.

**Step 1: Find all critical numbers**

$$f'(x) = 2x - 10$$

Only one critical number at $x = 5$. 
How do we determine if a critical point of a function $f$ gives a relative extremum?

We use the "**First Derivative Test**".

Let $f(x) = x^2 - 10x + 33$. Determine where $f$ is inc./dec. and find any relative extremum.

**Step 2: Test intervals:**

The critical point at $x = 5$ divides the domain of $f$ into two intervals:

- $f$ is decreasing on $(-\infty, 5)$
- $f$ is increasing on $(5, \infty)$

$$f'(4) = -2 < 0$$

$$f'(6) = 2 > 0$$
How do we determine if a critical point of a function $f$ gives a relative extremum?

We use the “**First Derivative Test**”.

Let $f(x) = x^2 - 10x + 33$. Determine where $f$ is inc./dec. and find any relative extremum.

**Step 3: Determine any relative extremum:**

- $f$ is **decreasing** on $(-\infty, 5)$
- $f$ is **increasing** on $(5, \infty)$

$f$ goes from **dec.** to **inc.** at $x = 5$  $\Rightarrow$  $f(5)$ is a **relative minimum**.
The graph of $f(x) = x^2 - 10x + 33$: 
Let $g(t) = t - \frac{1}{t}$.

Determine where $g$ is inc./dec. and find any relative extremum.

**Step 1: Find all critical points:**

$$g'(t) = 1 + \frac{1}{t^2}$$

$$1 + \frac{1}{t^2} = 0 \implies \frac{1}{t^2} = -1 \implies \text{No solution!}$$

$g'(t)$ is undefined at $t = 0$ which is **NOT** in the domain of $g$.

So there are **NO** critical points.
Relative Extremum & Critical Points

Let \( g(t) = t - \frac{1}{t} \).

Determine where \( g \) is inc./dec. and find any relative extremum.

\[
\left( g'(t) = 1 + \frac{1}{t^2} \right)
\]

Step 2: Test intervals

The domain of \( g \) is divided into two intervals:

- \( g \) is increasing on \((−∞, 0)\)
- \( g \) is increasing on \((0, ∞)\)

\[
\begin{align*}
g'(−1) &= 2 > 0 \\
g'(1) &= 2 > 0
\end{align*}
\]
Let \( g(t) = t - \frac{1}{t} \).

Determine where \( g \) is inc./dec. and find any relative extremum.

\[
\left( g'(t) = 1 + \frac{1}{t^2} \right)
\]

Step 2: Use the derivative test to determine intervals of inc./dec.

The domain of \( g \) is divided into two intervals:

- \( g \) is increasing on \((-\infty, 0)\)
- \( g \) is increasing on \((0, \infty)\)

\[
g'(-1) = 2 > 0 \quad \text{and} \quad g'(1) = 2 > 0
\]

So, \( g \) has NO relative extrema.
The graph of $g(t) = t - \frac{1}{t}$:
Relative Extremum & Critical Points

Let $f(x) = \sqrt{25 - x^2}$ for $-5 \leq x \leq 4$.

Determine where $f$ is inc./dec. and find any relative extremum.

Step 1: Find all critical points:

$$f'(x) = \frac{1}{2} (25 - x^2)^{-1/2} \cdot (-2x) = - \frac{x}{\sqrt{25 - x^2}}$$

$$\frac{x}{\sqrt{25 - x^2}} = 0 \implies x = 0$$

$f'(x)$ is undefined at $x = \pm 5$, but only $-5$ is in the domain of $f$.

So there are 2 critical points, $x = -5, 0.$
Let \( f(x) = \sqrt{25 - x^2} \) for \(-5 \leq x \leq 4\).

Determine where \( f \) is inc./dec. and find any relative extremum.

\[
\left( f'(x) = \frac{-x}{\sqrt{25 - x^2}} \right)
\]

**Step 2: Test intervals**

The domain of \( f \) is divided into two intervals:

\[
\begin{align*}
&f \text{ is increasing on } (-5, 0) \\
&f \text{ is decreasing on } (0, 4)
\end{align*}
\]

\[
\begin{align*}
f'(-1) &= \frac{1}{\sqrt{24}} > 0 \\
f'(1) &= -\frac{1}{\sqrt{26}} < 0
\end{align*}
\]

**relative max.**: 5 at \( x = 0 \)

**relative min.**: 0 at \( x = -5 \) and 3 at \( x = 4 \)
The graph of \( f(x) = \sqrt{25 - x^2} \):
How to find intervals where a function $f$ is increasing/decreasing and any relative extrema:

**Step 1:** *Find all critical points.*

**Step 2:** *Partition the domain of $f$ with the critical points.*

**Step 3:** *Test each interval.*

**Step 4:** *Determine relative extrema.*
Practice Problems
Find any relative extrema of the following functions:

(1) \( f(x) = x^3 - 3x^2 \) on \(-\infty \leq x \leq 3\)

(2) \( g(t) = t^2 + \frac{1}{t} \)

(3) \( h(x) = 3xe^x + 2 \)