ANTIDERIVATIVES

Section 4.8
November 14, 2013
The primary skill we’ve learned in this course so far is how to take the **derivative** of a function.

In Chapter 7, we’re going to do the “opposite”.

**Definition**

Let $f$ and $F$ be functions. If

$$F'(x) = f(x)$$

we say that $F$ is an **antiderivative** of $f$. 
Find an antiderivative of the following functions:

(1) \( f(x) = 6x \)  \hspace{1cm} (2) \( g(t) = 3e^{3t} \)  \hspace{1cm} (3) \( h(v) = \cos(2v) \)

(1) \( F(x) = 3x^2 \)  \hspace{1cm} (2) \( G(t) = e^{3t} \)  \hspace{1cm} (3) \( H(v) = \frac{1}{2} \sin(2v) \)

\[
\begin{align*}
F(x) &= 3x^2 + 1 \\
G(t) &= e^{3t} - 2 \\
H(v) &= \frac{1}{2} \sin(2v) + 3
\end{align*}
\]

\[
\begin{align*}
F(x) &= 3x^2 - 17 \\
G(t) &= e^{3t} + \pi \\
H(v) &= \frac{1}{2} \sin(2v) + e
\end{align*}
\]
**Antiderivatives**

\[ F \text{ is an antiderivative of } f \]

\[ \Downarrow \]

\[ F + C \text{ is an antiderivative of } f \text{ for any real number } C! \]

This follows from the fact that \( \frac{d}{dx}(C) = 0 \):

\[ \frac{d}{dx}(F(x) + C) = \frac{d}{dx}(F(x)) + \frac{d}{dx}(C) = \frac{d}{dx}(F(x)) + 0 = f \]

**Fact**

If \( F \) and \( G \) are any two antiderivatives of the function \( f \), then \( F - G = C \) for some real number \( C \).
THE INDEFINITE INTEGRAL

Definition
Let $f$ be a function. The **indefinite integral**

\[ \int f(x) \, dx \]

is the set of **ALL** antiderivatives of $f$. So, if $F$ is any antiderivative of $f$

\[ \int f(x) \, dx = \{ F(x) + C : C \text{ is a real number} \} = F(x) + C \]

for any real number $C$. In this definition:

\[ \int : \text{ integral symbol} \]

\[ f(x) : \text{ the integrand} \]
For example,

(1) If \( f(x) = 6x \), then
\[
\int f(x) \, dx = 3x^2 + C
\]

(2) If \( g(t) = 3e^{3t} \), then
\[
\int g(t) \, dt = e^{3t} + C
\]

(3) If \( h(v) = \cos(2v) \), then
\[
\int h(v) \, dv = \frac{1}{2} \sin(2v) + C
\]
Antidifferentiation Rules

Constant multiple rule

For any real number $k$,

$$\int k \cdot f(x) \, dx = k \int f(x) \, dx$$

if $\int f(x) \, dx$ exists.

Sum/Difference rule

$$\int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

if $\int f(x) \, dx$ and $\int g(x) \, dx$ exist.

Every indefinite integral that you will be asked find in this class will exist.
Antidifferentiation Rules

Power rule
For any real number $n \neq -1$

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

Basic Trig Functions
For any real number $k \neq 0$

$$\int \sin(kx) \, dx = -\frac{1}{k} \cos(kx) + C$$

$$\int \cos(kx) \, dx = \frac{1}{k} \sin(kx) + C$$
More Trig Functions

For any real number $k \neq 0$

\[
\int \sec^2(kx) \, dx = \frac{1}{k} \tan(kx) + C
\]

\[
\int \csc^2(kx) \, dx = -\frac{1}{k} \cot(kx) + C
\]

\[
\int \sec(kx) \tan(kx) \, dx = \frac{1}{k} \sec(kx) + C
\]

\[
\int \csc(kx) \cot(kx) \, dx = -\frac{1}{k} \csc(kx) + C
\]
Inverse Trig Functions

For any real number $k \neq 0$

\[
\int \frac{1}{\sqrt{1 - (kx)^2}} \, dx = \frac{1}{k} \sin^{-1}(kx) + C \quad (-1 < kx < 1)
\]

\[
\int \frac{1}{1 + (kx)^2} \, dx = \frac{1}{k} \tan^{-1}(kx) + C
\]

\[
\int \frac{1}{x \sqrt{(kx)^2 - 1}} \, dx = \sec^{-1}(kx) + C \quad (kx > 1)
\]
**Exponential Rule**

For **any** real number $k \neq 0$ and $b > 0$,

$$\int b^{kx} \, dx = \frac{b^{kx}}{\ln(b)k}$$

**Indefinite integral of $1/x$**

$$\int \frac{1}{x} \, dx = \ln|x| + C \quad (x \neq 0)$$

Why do we need that absolute value?
(1) \[ \int x^7 + \sin(5x) \, dx = \int x^7 \, dx + \int \sin(5x) \, dx = \frac{1}{8}x^8 - \frac{1}{5} \cos(5x) + C \]

(2) \[ \int -\csc^2(10t) + e^{-4t} \, dt \]

\[ = - \int \csc^2(10t) \, dt + \int e^{-4t} \, dt = \frac{1}{10} \cot(10t) - \frac{1}{4}e^{-4t} + C \]

(3) \[ \int \frac{1 + \sec(x/2) \tan(x/2)}{3} \, dx \]

\[ = \frac{1}{3} \left( \int 1 \, dx + \int \sec(x/2) \tan(x/2) \, dx \right) = \frac{1}{3} \left( x + 2 \sec(x/2) \right) + C \]

(4) \[ \int \frac{3}{x\sqrt{9x^2 - 1}} \, dx = \frac{1}{3} \int \frac{1}{x\sqrt{9x^2 - 1}} \, dx = 3 \sec^{-1}(3x) + C \]
Initial Value Problems
Initial Value Problems

$2x$ given a function
Initial Value Problems

\[ x^2 \quad x^2 + 5 \quad \cdots \quad x^2 + \pi \quad \cdots \]

\[ \int 2x \, dx \]

infinitely many antiderivatives

given a function
Initial Value Problems

\[ 2x^2 + x^2 + 5 \cdots x^2 + \pi \cdots \]

\[ \int 2x \, dx \]

specify a value

infinitely many antiderivatives

given a function
when \( x = 0 \), the antiderivative is 5
when $x = 0$, the antiderivative is 5

only one!

specify a value

infinitely many antiderivatives

given a function

\[ \int 2x \, dx \]
Suppose $f'(\theta) = \cos(\pi \theta)$ and $f(1/2) = 0$. What is $f$?

$$f(\theta) = \frac{1}{\pi} \sin(\pi \theta) + C \quad \text{for some real number } C.$$  

$$0 = f(1/2) = \frac{1}{\pi} \sin(\pi/2) + C \quad \implies \quad C = -\frac{1}{\pi}.$$  

$$f(\theta) = \frac{1}{\pi} \sin(\pi \theta) - \frac{1}{\pi}.$$
The **marginal profit** of the Chipotle on the corner of Dodge and 72nd, in thousands of dollars, is given by

\[ P'(b) = 3\sqrt{b + 1} \]

where \( b \) is the number of burritos sold (in thousands). If the profit is \(-2000\) when no burritos are sold, determine the profit function \( P(b) \).

\[
P(b) = 2(b + 1)^{3/2} + C
\]

\[
P(0) = -2000 \implies 2(0 + 1)^{3/2} + C = -2000
\]

\[
\implies C = -2002
\]

\[
\implies P(b) = 2(b + 1)^{3/2} - 2002
\]
An object is dropped from an height of $h_0$ with an initial velocity of $v_0$. If the acceleration due to the Earth’s gravity is $-16$ ft./sec$^2$, show that the height of the object at time $t$ is given by

$$h(t) = -8t^2 + v_0 t + h_0$$

$$h''(t) = -16 \implies h'(t) = -16t + C \text{ for some } C$$

Initial velocity is $v_0$,

$$v_0 = h'(0) = -16(0) + C = C \implies h'(t) = -16t + v_0$$

$$h'(t) = -16t + v_0 \implies h(t) = -8t^2 + v_0 t + C \text{ for some } C$$

Initial height is $h_0$,

$$h_0 = h(0) = C \implies h(t) = -8t^2 + v_0 t + h_0$$