AREA BETWEEN CURVES

We were first introduced to the definite integral of a function over an interval as having something to do with the area between the graph of the function and the x-axis over that interval. In this section we want to better understand this. Since determining definite integrals is central to doing this, let’s refresh our memory on how to determine definite integrals by looking at some examples.

1.1 SOME EXAMPLES OF DEFINITE INTEGRALS

Let’s start by determining the following definite integral:

\[
\int_0^1 x\sqrt{1-x^2} \, dx
\]

By the second part of the Fundamental Theorem of Calculus (FTC), we know that we can determine this definite integral by first finding an antiderivative \( F \) of \( x\sqrt{1-x^2} \) (any antiderivative will do) and then computing the difference \( F(1) - F(0) \).

So, let’s find an \( F \). Recall that the indefinite integral

\[
\int x\sqrt{1-x^2} \, dx
\]

gives all antiderivatives of \( x\sqrt{1-x^2} \), so by specifying a value of \( C \) (\( C = 0 \) would work nicely) we’ll have our \( F \).

Now, we don’t have an antiderivative formula for \( x\sqrt{1-x^2} \) so in order to determine the above indefinite integral we’ll need to use substitution. If we think back to the general guidelines for choosing our \( u \), we remember that one promising choice is anything under a root. So let

\[
u = 1 - x^2
\]

This means that

\[
\begin{align*}
\text{du} &= -2x \, dx \\
\text{dx} &= -\text{du}/2x
\end{align*}
\]

Plugging all of this into our indefinite integral we get
\[
\int x\sqrt{1-x^2} \, dx = \int x\sqrt{u} \, -\frac{du}{2x} = -\frac{1}{2} \int \sqrt{u} \, du
\]

We know how to determine this indefinite integral, so we know we made the good choice for \( u \). Continuing on...

\[-\frac{1}{2} \int \sqrt{u} \, du = -\frac{1}{3} u^{3/2} + C = -\frac{1}{3} (1-x^2)^{3/2} + C.\]

Setting \( C = 0 \), we have the antiderivative

\[ F(x) = -\frac{1}{3} (1-x^2)^{3/2}. \]

So, by the FTC part II

\[
\int_0^1 x\sqrt{1-x^2} \, dx = F(1) - F(0) = 0 - \left( -\frac{1}{3} \right) = \frac{1}{3}.
\]

Now, what we just did is perfectly correct and will always work, but we actually did a little more work than we needed to. Suppose we wanted to perform our \( u \)-substitution with the definite integral rather than doing it the indefinite integral, that is

\[
\int_0^1 x\sqrt{1-x^2} \, dx = \frac{1}{2} \int_?^? \sqrt{u} \, du.
\]

Why the ?'s on the integral sign? Well, the definite integral on the left is taken with respect to \( x \) over the interval from \( x = 0 \) to \( x = 1 \). The definite integral on the right is taken with respect to the variable \( u = 1-x^2 \), so if \( x \) goes from \( x = 0 \) to \( x = 1 \) we know that \( u \) will go from \( (0)^2 = 1 \) to \( (1)^2 = 0 \). This means that

\[
\int_0^1 x\sqrt{1-x^2} \, dx = -\frac{1}{2} \int_1^0 \sqrt{u} \, du
\]

(the fact that the limits of integration flipped is just a coincidence). Let’s go ahead and determine this new definite integral

\[
-\frac{1}{2} \int_1^0 \sqrt{u} \, du = -\frac{1}{3} u^{2/3} \bigg|_1^0 = -\frac{1}{3} (0)^{2/3} - \left( -\frac{1}{3} (1)^{2/3} \right) = \frac{1}{3}.
\]

This is same answer that we got doing it the other way! In general we have the following formula for substitution with definite integrals.

| Suppose \( f \) is continuous on the interval \([a, b]\). If you perform a \( u \)-substitution with \( u = g(x) \), and after substitution \( f(x) \, dx \) becomes \( h(u) \, du \) then |
|\[
\int_a^b f(x) \, dx = \int_{g(a)}^{g(b)} h(u) \, du. \] |
## Practice Problems

Determine the following definite integrals:

1. \[ \int_{2\pi}^{3\pi} 3 \cos^2(x) \sin(x) \, dx \]

2. \[ \int_{0}^{\sqrt{3}} \frac{4x}{\sqrt{x^2 + 1}} \, dx \]

3. \[ \int_{2}^{4} \frac{1}{x \ln(x)} \, dx \]

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### Solution

1. Let
   \[ u = \cos(x) \]
   \[ du = -\sin(x) \, dx \]
   \[ dx = -du/\sin(x) \]

   Then
   \[
   \int_{2\pi}^{3\pi} 3 \cos^2(x) \sin(x) \, dx = \int_{\cos(2\pi)}^{\cos(3\pi)} 3u^2 \sin(x) \left( \frac{-1}{\sin(x)} \right) \, du \\
   = -3 \int_{1}^{1} u^2 \, du = -u^3 \bigg|_{1}^{1} = -(-1)^3 + (1)^3 = 2
   
   \]

2. Let
   \[ u = x^2 + 1 \]
   \[ du = 2x \, dx \]
   \[ dx = du/2x \]

   Then
   \[
   \int_{0}^{\sqrt{3}} \frac{4x}{\sqrt{x^2 + 1}} \, dx = \int_{1}^{4} \frac{4x}{\sqrt{u}} \frac{1}{2x} \, du = 2 \int_{1}^{4} u^{-1/2} \, du \\
   = 4u^{1/2} \bigg|_{1}^{4} = 4(4)^{1/2} - 4(1)^{1/2} = 4
   
   \]

3. Let
   \[ u = \ln(x) \]
   \[ du = \frac{1}{x} \, dx \]
   \[ dx = x \, du \]

   Then
\[
\int_2^4 \frac{1}{x \ln(x)} \, dx = \int_{\ln(2)}^{\ln(4)} \frac{1}{xu} \, du = \int_{\ln(2)}^{\ln(4)} \frac{1}{u} \, du \\
= \ln |u| \bigg|_{\ln(2)}^{\ln(4)} = \ln \left| \frac{\ln(4)}{\ln(2)} \right| \approx 0.69
\]

### 1.2 The Area Between Curves

Suppose we want to determine the area between the graph of \( f(x) = \sin(x) \) and the \( x \)-axis over the interval \([0, 3\pi/2]\), i.e. the shaded region below

![Graph of \( y = \sin(x) \)]

Note that since the graph of \( y = \sin(x) \) is both above and below the \( x \)-axis on the interval \([0, 3\pi/2]\) we can’t just take the definite integral since

\[
\int_0^{3\pi/2} \sin(x) \, dx = \text{“shaded area above the } x \text{-axis”} - \text{“shaded area below the } x \text{-axis”}
\]

So how do we determine area of the shaded region? Well, the shaded area with be the definite integral over the intervals where \( f(x) = \sin(x) \) is positive plus the absolute value of the definite integral over the intervals where \( f(x) = \sin(x) \) is negative.

The only time \( \sin(x) = 0 \) in the interval \([0, 3\pi/2]\) is when \( x = \pi \). And we can see that \( \sin(x) \) is positive on the interval \((0, \pi)\) and negative on the interval \((\pi, 3\pi/2)\). So the area of the shaded region is

\[
\int_0^{\pi} \sin(x) \, dx + \left| \int_{\pi}^{3\pi/2} \sin(x) \, dx \right| = -\cos \left. \frac{\pi}{\pi} \right|_0^{\pi} + \left| -\cos \left. \frac{3\pi/2}{\pi} \right|_\pi^{3\pi/2} \right| \\
= -\cos(\pi) + \cos(0) + | -\cos(3\pi/2) + \cos(\pi) | = 1 + 1 + |0 - 1| = 3.
\]

So, to determine the area between the graph of a function and the \( x \)-axis we do the following
Let's look at an example:

**Example:** Determine the area between the graph of \( f(x) = x^3 - 4x^2 + 3x \) and the x-axis over the interval \([0,3]\).

First we want to determine the zeros of \( f \) in the interval \([0,3]\).

\[
x^3 - 4x^2 + 3x = 0 \\
x(x^2 - 4x + 3) = 0 \\
x(x - 1)(x - 3) = 0
\]

So \( f \) has three zeros, but only one is inside the interval \([0,3]\), namely \( x = 1 \), the others are actually the endpoints of the interval. These zeros subdivide the interval \([0,3]\) into two subintervals \([0,1]\) and \([1,3]\).

Next, we integrate over each subinterval

\[
\int_0^1 (x^3 - 4x^2 + 3x) \, dx = \frac{1}{4} x^4 - \frac{4}{3} x^3 + \frac{3}{2} x^2 \bigg|_0^1 = \frac{5}{12}
\]

\[
\int_1^3 (x^3 - 4x^2 + 3x) \, dx = \frac{1}{4} x^4 - \frac{4}{3} x^3 + \frac{3}{2} x^2 \bigg|_1^3 = -\frac{32}{12}
\]

Finally, we sum up the absolute values of the definite integrals above

\[
\left| \frac{5}{12} + \left| -\frac{32}{12} \right| \right| = \frac{37}{12}.
\]

Instead of thinking of what we’ve just done as determining the area between the graph of \( f \) and the x-axis over a given interval, we could think of it as finding the area between the graph of \( f \) and the graph of the function \( g(x) = 0 \) over the interval. From this perspective it is natural to ask if we can determine the area of a region defined between the graphs of two functions. The answer is “Yes” and we can do so by essentially the same process.
Let's look at an example:

**Example:** Determine the area between the graphs of \( f(x) = x^2 - 4 \) and \( g(x) = -x^2 - 2x \) over the interval \([-3, 1]\).

\[
\begin{align*}
2x^2 + 2x - 4 &= 0 \\
x^2 + x - 2 &= 0 \\
(x - 1)(x + 2) &= 0
\end{align*}
\]

So \( f - g \) has two zeros, \( x = 1 \) and \( x = -2 \). These zeros subdivide the interval \([-3, 1]\) into two subintervals \([-3, -2]\) and \([-2, 1]\).

Next, we integrate over each subinterval

\[
\begin{align*}
\int_{-3}^{-2} 2x^2 + 2x - 4 \, dx &= \frac{2}{3}x^3 + x^2 - 4x \bigg|_{-3}^{-2} = \frac{20}{3} - 3 = \frac{11}{3} \\
\int_{-2}^{1} 2x^2 + 2x - 4 \, dx &= \frac{2}{3}x^3 + x^2 - 4x \bigg|_{-2}^{1} = -\frac{7}{3} - \frac{20}{3} = -9
\end{align*}
\]

Finally, we sum up the absolute values of the definite integrals above

\[
\frac{11}{3} + |-9| = \frac{38}{3}.
\]
1.3 Practice Problems

(1) Determine the area of the region enclosed by the curves

(a) \( y = 5x - x^2 \) and \( y = x \) over the interval \([0, 4]\).
(b) \( y = 4 - x^2 \) and \( y = -x + 2 \) over the interval \([-2, 3]\).

(1a) First we want to determine the zeros of \( 5x - x^2 - x \) in the interval \([0, 4]\).

\[
\begin{align*}
4x - x^2 &= 0 \\
x(4-x) &= 0
\end{align*}
\]

So \( f - g \) has no zeros in the interval \([0, 4]\). So in this case all we have to do is integrate over \([0, 4]\)

\[
\int_0^4 4x - x^2 \, dx = 2x^2 - \frac{1}{3}x^3 \bigg|_0^4 = 32 \cdot \frac{64}{3} - 0 = \frac{32}{3}.
\]

(1b) First we want to determine the zeros of \( 4 - x^2 - (-x + 2) \) in the interval \([-2, 3]\).

\[
\begin{align*}
2 + x - x^2 &= 0 \\
(2-x)(1+x) &= 0
\end{align*}
\]

So \( f - g \) has two zeros, \( x = -1 \) and \( x = 2 \). These zeros subdivide the interval \([-2, 3]\) into three subintervals \([-2, -1]\), \([-1, 2]\) and \([2, 3]\).

Next, we integrate over each subinterval

\[
\begin{align*}
\int_{-2}^{-1} 2 + x - x^2 \, dx &= 2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \bigg|_{-2}^{-1} = \frac{7}{6} - \frac{2}{3} = -\frac{11}{6} \\
\int_{-1}^{2} 2x^2 + 2x - 4 \, dx &= 2x^3 + x^2 - 4x \bigg|_{-1}^{2} = \frac{10}{3} + \frac{7}{6} = \frac{27}{6} \\
\int_{2}^{3} 2x^2 + 2x - 4 \, dx &= 2x^3 + x^2 - 4x \bigg|_{2}^{3} = \frac{3}{2} - \frac{10}{3} = -\frac{11}{6}
\end{align*}
\]

Finally, we sum up the absolute values of the definite integrals above

\[
\frac{11}{6} + \frac{27}{6} + \frac{11}{6} = \frac{49}{6}.
\]