Differential Geometry Notes

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These notes were typed as part of my preparation for the Geometry/Topology qualifying exam in August 2005 (with a few later additions). Most of this material is at least relevant to the qualifying exam, though some of it may not actually be useful for solving qual problems. All of the results listed here are standard and should be easy to find in any differential geometry textbook. In fact, many of the theorems are copied word for word from Spivak's book; my citations are probably woefully inadequate. I have added some explanations of my own as well as several examples. The emphasis is on material that I did not understand well when I first encountered the subject and on specific results that were required to solve problems from previous quals. A fair amount of other material was added for completeness (and in some cases, out of compulsiveness). I hope that these notes will be useful to anyone studying for the qualifying exams, but my primary goal in typing them was to learn the material myself. Comments and corrections are welcome.

Differentiation on Euclidean Spaces

1. **The Inverse Function Theorem:** (Munkres 62-70)
Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable. For f to have a differentiable inverse, it is necessary that Df be nonsingular. (This follows from the chain rule.) The inverse function theorem says that, locally, this is also a sufficient condition. Here is the precise statement.
Theorem (The Inverse Function Theorem). Let A be open in $\mathbb{R}^n$, and let $f : A \longrightarrow \mathbb{R}^n$ be of class C^r. If Df is nonsingular at $a \in A$, there is a neighborhood U of a such that f carries U in a one-to-one fashion onto an open set $V \subset \mathbb{R}^n$ and the inverse function is of class C^r.

2. **The Implicit Function Theorem:** (Garrity 56-59)
Suppose you have a manifold M defined as the zero locus of a set of k functions $\{f_j : \mathbb{R}^{n+k} \longrightarrow \mathbb{R} \mid j = 1, \ldots, k\}$. That is, using the labels $x_1, \ldots, x_n, y_1, \ldots, y_k$ for a coordinate system on $\mathbb{R}^{n+k}$ and using the abbreviations

$$x = x_1, \ldots, x_n$$
$$y = y_1, \ldots, y_k,$$

$$M = \{(x, y) \in \mathbb{R}^{n+k} \mid f_1(x, y) = 0, \ldots, f_k(x, y) = 0\}.$$

We want to know when, given a point $(a, b) \in M \subset \mathbb{R}^{n+k}$, there are k functions $p_1, \ldots, p_k$ such that for all points (x, y) in a neighborhood $V \subset M$ of (a, b),

$$p_1(x) = y_1, \ldots, p_k(x) = y_k.$$

In other words, for all $(x, y) \in V$,

$$f_1(x, p_1(x)) = 0, \ldots, f_k(x, p_k(x)) = 0.$$

The implicit function theorem gives a condition (which is easy to compute) under which, for a given $(a, b) \in M$, the functions $p_1, \ldots, p_k$ exist.
Theorem (The Implicit Function Theorem). Let $f_1, \ldots, f_k$ be C^r functions on $\mathbb{R}^{n+k}$, and suppose that $(a, b) \in \mathbb{R}^{n+k}$ is a point such that $f_j(a, b) = 0$ for $j = 1, \ldots, k$. Suppose that at the point (a, b), the $k \times k$ matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial y_1} & \cdots & \frac{\partial f_k}{\partial y_k} \end{pmatrix}$$

is invertible. Then in a neighborhood of $a \in \mathbb{R}^n$, there are k unique C^r functions

$$p_1, \ldots, p_k$$

such that

$$f_1(x, p_1(x)) = 0, \ldots, f_k(x, p_k(x)) = 0.$$

Here is a simple example. Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f(x, y) = x^2 + y^2 - 1,$$

and let M be the zero locus of f. So $M = S^1$. In this case, $k = 1$ and the matrix from the theorem is

$$\frac{\partial f}{\partial y} = 2y.$$

This matrix is invertible whenever $y \neq 0$. So y can be written as a function of x at every point of S^1 except $(1, 0)$ and $(-1, 0)$. Of course, in this case, we know that if $y > 0$, then $y = \sqrt{1 - x^2}$ and if $y < 0$, $y = -\sqrt{1 - x^2}$.

3. The implicit function theorem tells you whether or not some of the variables can be written in differentiable terms of the other variables when restricted to the zero locus of some set of functions. In the two-variable case, it tells you when y can be written as a differentiable function of x or vice-versa. It can happen that y is a function of x even when the relevant derivative is 0. It just won't be a *differentiable* function of x. The same kind of thing can happen with the inverse function theorem.
   Example: Let

$$h(x, y) = y - \sqrt[3]{x}.$$

   Then $\frac{\partial h}{\partial x}(0, 0) = 0$. Yet $y = \sqrt[3]{x}$ on the zero set of h. So y is a function of x, but this function is not differentiable at 0.

Topological Manifolds

1. If you have a manifold embedded in $\mathbb{R}^n$, you have to be careful about using the word "coordinates." If the manifold has dimension less than n, the $\mathbb{R}^n$ coordinates of the manifold (the coordinates of the ambient space) do not constitute a coordinate chart. For example, S^2 is embedded in $\mathbb{R}^3$, but S^2 is a 2-manifold. xyz-coordinates are not a global chart on S^2 because S^2 is not *open* in $\mathbb{R}^3$. You need to use coordinates with *two* components, such as the coordinates of stereographic projection (see below). However, in some situations, you can use ambient coordinates. For example, if you want to push a vector field forward from, say, the manifold $\mathbb{R}^2$ to S^2 via inverse stereographic projection, you can use the map φ_*^{-1} (defined below). This is a map from the tangent bundle of $\mathbb{R}^2$ to the tangent bundle of $\mathbb{R}^3$, but from the definition of the tangent space given below, it is obvious that the image of φ_*^{-1} is contained in TS^2.

2. **Theorem (Invariance of Domain).** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is one-to-one and continuous, then f is open.
   This theorem tells us, among other things, that a point in a manifold with boundary cannot have neighborhoods homeomorphic to both $\mathbb{R}^n$ and $\mathbb{H}^n$ (closed half-space). In other words, a point is either on the boundary or in the interior of the manifold, not both.

Smooth Manifolds

1. If you are given a set in $\mathbb{R}^n$ defined by an equation, one way to prove that it's a smooth manifold is to give a set of C^∞ related charts that cover the set. Here is another way:
   Definition. Let M and N be smooth manifolds and let $f : M \longrightarrow N$ be C^∞. If $p \in M$, $q = f(p)$, (U, x) and (V, y) are neighborhoods and charts around p and q, respectively, then the *rank* of f is defined to be the rank of the matrix

   $$\left(\frac{\partial (y^i \circ f)}{\partial x^j} \right).$$

   Theorem (Spivak 49). Suppose M and N are smooth manifolds of dimension m and n, respectively, and $f : M \longrightarrow N$ is smooth. For any $q \in N$, if f has constant rank k in a neighborhood of $f^{-1}(q)$, then $f^{-1}(q)$ is an $m-k$-dimensional closed submanifold of M (or is empty). In particular, if q is a regular value of f, then $f^{-1}(q)$ is an $m-n$-dimensional submanifold of M (or is empty). Example: Define a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

   $$f(x, y) = x^2 + y^2 - 1.$$

   Then

   $$Df = [2x \;\; 2y],$$

   and the rank of Df is 1 at every point except $(0, 0)$. It follows from the theorem that $f^{-1}(0) = S^1$ is a 1-dimensional closed submanifold of $\mathbb{R}^2$. We knew this already, but this theorem can be used in more complicated situations in which the result is not so easy to see otherwise.

2. **Lagrange Multipliers:**
   Theorem. Suppose M is a k-dimensional submanifold of $\mathbb{R}^n$ defined as the zero set of a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$. That is, $M = g^{-1}(0 \in \mathbb{R}^{n-k})$. Let $g(x_1, \ldots, x_n) = (g_1(x_1, \ldots, x_n), \ldots, g_{n-k}(x_1, \ldots, x_n))$ If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth, then $p \in M$ is a critical point of the restriction of f to M if and only if there exist real constants $\lambda_1, \ldots, \lambda_{n-k}$ (called "Lagrange Multipliers") such that

   $$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_{n-k} \nabla g_{n-k}.$$

   $\mathbb{R}$, and f is the restriction of this map to S^2.) To find the critical points of f, first observe that S^2 is the zero set of the function

   $$g : \mathbb{R}^3 \longrightarrow \mathbb{R} : (x, y, z) \mapsto (x^2 + y^2 + z^2 - 1).$$

3. Important Fact: If M is a manifold embedded in $\mathbb{R}^n$ which consists of all points $(x_1, \ldots, x_n)$ such that $f(x_1, \ldots, x_n) = 0$, then ∇F =Grad(f) is normal to M.

Tangent Bundles

1. The *tangent space* at a point p of a smooth manifold M can be defined as the set of equivalence classes of curves in M running through p under the following equivalence relation: Given a coordinate x in a neighborhood of p, $\rho \sim \gamma$ iff $(x \circ \rho)'(0) = (x \circ \gamma)'(0)$. The tangent space at p is denoted by T_pM.

2. For each $p \in M$, T_pM is a vector space of dimension n, where n is the dimension of M.

3. The set of all tangent spaces for a smooth manifold is called the *tangent bundle*. A smooth manifold together with its tangent bundle (and the obvious projection map) make a vector bundle, a more general kind of object. General vector bundles are not important for the quals.

4. Given a C^∞ map f from M to N, at each point $p \in M$, there is a linear function from T_pM to T_pN denoted by f_{p*}. If f is one-to-one near p (i.e., if f has full rank in a neighborhood of p), then f_{p*} is a vector space isomorphism. In local coordinates, f can be thought of as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$, and f_{p*} is defined by

   $$v_p \mapsto [Df_p(v)]_{f(p)}.$$

   That is to say, the vector v sitting at $p \in \mathbb{R}^m$ gets mapped by the Jacobian of f at p to the vector $Df_p(v)$, which is sitting at the point $f(p) \in \mathbb{R}^n$.

   In the particular case where M and N are *embedded* in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, if f is written in $\mathbb{R}^m$ and $\mathbb{R}^n$ coordinates, f_{p*} can be defined exactly as above. In this case, the vector v sitting at $p \in M \subset \mathbb{R}^m$ gets mapped by the Jacobian of f at p to the vector $Df_p(v)$, which is sitting at the point $f(p) \in N \subset \mathbb{R}^n$ This is confusing, but it works because f maps curves in M to curves in N. Therefore, the image of the tangent vector to a curve γ in M under Df (written in ambient coordinates) is necessarily tangent to the image of γ under f.

5. The union over all $p \in M$ of f_{p*} is a map from TM to TN, which we denote by f_*. If f is a diffeomorphism, then a vector field V on M can be pushed forward via f_* to a vector field on N. More precisely, if V is a vector field on M, then we can define a vector field W on N by

   $$W(p) = f_*(V(f^{-1}(p))).$$

   Clearly, this does not make sense if f is not one-to-one.

Multilinear Algebra

1. An *alternating k-tensor* on a vector space V is a multilinear map ω from $V^k = V \times V \times \cdots \times V$ (the cartesian product of k copies of V) to $\mathbb{R}$ which changes sign when two of the arguments are interchanged. That is, $\omega(v^1, \ldots, v^i, \ldots, v^j, \ldots, v^k) = -\omega(v^1, \ldots, v^j, \ldots, v^i, \ldots, v^k)$.

2. Notation: $\Omega^k(V)$ denotes the set of alternating k-tensors on the vector space V.

3. A k-form or differential form on a manifold, M, is a function that for each point $p \in M$ assigns an alternating k-tensor to T_pM.

4. Notation: If ω is a k-form on M and $p \in M$, then w_p denotes the alternating k-tensor that ω assigns to T_pM.

5. What is dx^i? If M is an n-manifold and $U \subset M$ has local coordinates $x^1, \ldots, x^n$, then for every point $p \in U$, dx^i denotes the unique alternating 1-tensor (i.e., linear functional) on T_pM such that $dx^i(\frac{\partial}{\partial x^j}) = \delta_{ij}$. An alternative (but equivalent) definition that is often more useful for computational purposes is the following:

$$dx^i(a_1, \ldots, a_n) = a_i.$$

   That is, dx^i just picks out the ith component of a vector.

6. It can be proved that, locally, any k-form ω on M can be written in the form

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

   where the $\omega_{i_1,\ldots,i_k}$'s are functions from M to $\mathbb{R}$.

7. The exterior derivative of a k-form, ω is a (k+1)-form, denoted by $d\omega$.

8. Some properties of the exterior derivative:

   a. $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$

   b. If ω_1 is a k-form, and ω_2 is an m form, then

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2.$$

   c. $d(d\omega) = 0$.

   d. If $f : M \longrightarrow N$ is smooth and ω is a form on N, then $d(f^*\omega) = f^* d\omega$.

9. Given a vector $v \in T_pM$ and an alternating k-tensor ω on T_pM, the *interior product* of ω with v is an alternating $k-1$ tensor, denoted by $i_v\omega$, which is defined by

$$i_v\omega(v_1, \ldots, v_{k-1}) := \omega(v, v_1, \ldots, v_{k-1}).$$

   If V is a vector field and ω is a form on M, then we can define a $k-1$ form $i_V\omega$ by

$$i_V\omega_p(v_1, \ldots, v_{k-1}) := \omega_p(V_p, v_1, \ldots, v_{k-1}).$$

   (Here, $p \in M$ and $v_1, \ldots, v_{k-1} \in T_pM$.)

   $i_v\omega$ is sometimes called the *contraction* of ω in the direction of v.

10. The following observation can be used to make evaluating forms easier. For a given point $p \in M$, if $v^1, \ldots, v^n$ are vectors in T_pM, then

$$(dx^1 \wedge \cdots \wedge dx^n)(v^1, \ldots, v^n) = \det \begin{pmatrix} dx^1(v^1) & \ldots & dx^n(v^1) \\ \vdots & & \vdots \\ dx^1(v^n) & \ldots & dx^n(v^n) \end{pmatrix}.$$

This is easy to remember because the matrix has exactly the same form as the Jacobian of a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Just replace the components of the function with $v^1, \ldots, v^n$, and replace the partial derivatives $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ with $dx^1, \ldots, dx^n$.

11. Here is a sample computation: Let $\omega = x^3 y dx \wedge dy$ be a 2-form on the manifold $M = \mathbb{R}^2$, let $p = (1, 2)$ be a point in $\mathbb{R}^2$, and let $v^1 = (3, 4)$ and $v^2 = (5, 6)$ be vectors in T_pM. Then

$$\omega_p(v^1, v^2) = 1^3 \cdot 2 \cdot \det \begin{pmatrix} dx(v^1) & dy(v^1) \\ dx(v^2) & dy(v^2) \end{pmatrix} = 2 \cdot \det \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = 2(18 - 20) = -4.$$

12. The determinant formula above is also useful for computing interior products. For example, let $V = (xz, y^3 x, x^7)$ be a vector field on $\mathbb{R}^3$. Then

$$(i_V dx \wedge dy \wedge dz)(u^1, u^2) = dx \wedge dy \wedge dz(V, u^1, u^2) = \det \begin{pmatrix} dx(V) & dy(V) & dz(V) \\ dx(u^1) & dy(u^1) & dz(u^1) \\ dx(u^2) & dy(u^2) & dz(u^2) \end{pmatrix} =$$

$$\det \begin{pmatrix} xz & y^3 x & x^7 \\ dx(u^1) & dy(u^1) & dz(u^1) \\ dx(u^2) & dy(u^2) & dz(u^2) \end{pmatrix} = xz dy \wedge dz(u^1, u^2) - y^3 x dx \wedge dz(u^1, u^2) + x^7 dx \wedge dy(u^1, u^2).$$

That is,

$$i_V dx^1 \wedge dx^2 \wedge dx^3 = xz dy \wedge dz - y^3 x dx \wedge dz + x^7 dx \wedge dy.$$

13. If

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

in local coordinates, then

$$d\omega = \sum_{i_1 < \cdots < i_k} d\omega_{i_1, \ldots, i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

14. A form ω is called *closed* if $d\omega = 0$.

15. A k-form ω is called *exact* if there exists a $k - 1$-form η such that $d\eta = \omega$.

16. **Theorem (the Poincaré Lemma).** If M is smoothly contractible to a point $p \in M$, then every closed form on M is exact.

17. If $k > n$, then any k-form on an n-manifold is zero. This is true because any set of k vectors in an n-dimensional vector space is linearly dependent. Thus, any n-form on an n-manifold is closed.

18. Pullback of a form: If $f : M \longrightarrow N$ is smooth, and ω is a k-form on N, then we can define a form $f^*\omega$ on M by

$$f^*\omega(v_1, \dots, v_k) := \omega(f_*(v_1), \dots, f_*(v_k)).$$

The form $f^*\omega$ is called the "pullback" of ω. If x and y are coordinate systems on M and N, respectively, then the following rules can be used to compute $f^*\omega$.

a. $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$

b. $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$

c.

$$f^*(dy^i) = \sum_{j=1}^{n} \frac{\partial(y^i \circ f)}{\partial x^j} dx^j.$$

d. If g is a 0-form, then $f^*g = g \circ f$.

Note: I'm using the convention that for a 0-form (a smooth function on M) g,

$$g \wedge \omega = g \cdot \omega.$$

Example: Let $M = N = \{(x, y) \in \mathbb{R} \,|\, x, y > 0\}$ and let f be the identity map. Let (x, y) and (u, v) be coordinate charts on M where $u = x^2$ and $v = 2x + y$. That is,

$$f(x, y) = (x^2, 2x + y) = (u, v),$$

and

$$f^{-1}(u, v) = (\sqrt{u}, v - 2\sqrt{u}) = (x, y).$$

Let $\omega = xy^2 dx \wedge dy$. Then

$$\begin{aligned}
&f^*\omega = f^*(xy^2)f^*(dx) \wedge f^*(dy) = (x \circ f)(y \circ f)^2 d(x \circ f) \wedge d(y \circ f) \\
&= (\sqrt{u})(v - 2\sqrt{u})^2 d(\sqrt{u}) \wedge d(v - 2\sqrt{u}) = (\sqrt{u})(v - 2\sqrt{u})^2 \left(-\frac{1}{2\sqrt{u}} du\right) \wedge (dv - \left(\frac{1}{\sqrt{u}}\right) du) \\
&= (\sqrt{u})(v - 2\sqrt{u})^2 \left(-\frac{1}{2\sqrt{u}}\right) \left(-\frac{1}{\sqrt{u}}\right) du \wedge dv = \left(\frac{v\sqrt{u} - 2u)^2}{2u}\right) du \wedge dv.
\end{aligned}$$

19. Pullback of a top-degree form:
Computing pullbacks can be tedious, but there is a formula for pulling back top-degree forms that makes things marginally less painful (although, to be honest, I never use this formula because I can never remember it).
Theorem (Spivak 208). If $f : M \longrightarrow N$ is a C^∞ function between n-manifolds, x is a chart on a neighborhood U of $p \in M$, and y is coordinate chart on a neighborhood V of $q = f(p) \in N$, then

$$f^*(g dy^1 \wedge \cdots \wedge dy^n) = (g \circ f) \cdot \det \left(\frac{\partial(y^i \circ f)}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n.$$

It is often useful to switch from one coordinate chart to another on the same manifold (e.g., from rectangular to polar coordinates on $\mathbb{R}^2 \setminus \{\text{a ray starting at the origin}\}$). The theorem above can be used to simplify this computation by setting $N = M$ and $f =$ the identity map on M.

Vector Fields

Most of the following is summarized from (Spivak Chapter 5).

1. A vector field on a manifold M is a function from M to TM. We usually assume that vector fields are smooth. The value of a vector field X at a point $p \in M$ is often denoted by X_p or $X(p)$.

2. In local coordinates, a vector field can be written as

$$X = (g_1(x^1, \ldots, x^n), \ldots, g_n(x^1, \ldots, x^n))$$

or as

$$X = \sum_{i=1}^{n} g_i(x^1, \ldots, x^n) \left(\frac{\partial}{\partial x^i} \right).$$

The latter notation comes from the fact that the tangent space at a point $p \in M$ can be defined as the set of derivations at p.

3. For a vector field X and a point $p \in M$, an *integral curve* for X is a function

$$c : [0,1] \longrightarrow M$$

such that $c(0) = p$ and $c'(t) = X_c(t)$ for all $t \in [0,1]$. Intuitively, if X is pictured as a series of arrows on M, an integral curve at p is a curve in M that passes through p and goes in the direction of the arrows at every point.

4. **Theorem (Spivak 147).** If X is a C^∞ vector field on M and $p \in M$, then there is an open set V containing p and an $\varepsilon > 0$ such that there is a unique collection of diffeomorphisms $\{\phi_t\} : V \longrightarrow \phi_t(V) \subset M$ for $|t| < \varepsilon$ with the following properties:

   a. $\phi : (-\varepsilon, \varepsilon) \longrightarrow M$, defined by $\phi(t,p) = \phi_t(p)$, is C^∞.

   b. If $|s|, |t|, |s+t| < \varepsilon$ and $q, \phi_t(q) \in V$, then

$$\phi_{s+t}(q) = (\phi_s \circ \phi_t)(q).$$

   c. If $q \in V$, then X_q is the tangent vector at $t = 0$ to the curve $t \mapsto \phi_t(q)$.

The family $\{\phi_t\}$ is called a *(local) 1-parameter group of diffeomorphisms* generated by X. The group operation is composition. The vector field X is sometimes called the "infinitesimal generator" of $\{\phi_t\}$, though this terminology doesn't mean anything to me.

5. The function ϕ is also called a *local flow* for X on V. A local flow for X on V is really just a collection of integral curves for X that pass through every point in V.

6. **Theorem (Spivak 148).** Let X be a C^∞ vector field on M such that $X_p \neq 0$. Then there is a coordinate system x on a neighborhood U of p such that $X = \frac{\partial}{\partial x^1}$ on U.

7. Given a vector field

$$X = \sum_{i=1}^{n} g_i(x^1, \ldots, x^n) \left(\frac{\partial}{\partial x^i} \right)$$

on M and a C^∞ function $f : M \longrightarrow \mathbb{R}$, X acts on f by

$$Xf(p) := \left(\sum_{i=1}^{n} g_i(p) \left(\frac{\partial f}{\partial x^i} \right) \right).$$

In other words, the value of Xf at p is the directional derivative of f in the direction of X_p.

8. Now we want to define the rates of change of three kinds of objects in the direction of X. These rates of change will be called *Lie derivatives*. They are all modelled after the definition of a directional derivative in Euclidean space. For the following, let X be a vector field on M, let p be a point in M, and let ϕ be a local flow for X around p.

   a. If f is a C^∞ function on M, then the Lie derivative of f with respect to X is

   $$L_X f := Xf.$$

   So this is just a new notation. The reason for using this notation is to emphasize that the following two definitions are very similar.

   b. Given another vector field Y on M, the Lie derivative of Y with respect to X is defined by

   $$L_X Y(p) := \lim_{h \to 0} \frac{1}{h} \left[Y_p - (\phi_{h*} Y)_p \right].$$

   $L_X Y$ is a new vector field on M. Intuitively, $L_X Y$ as the instantaneous rate of change of Y in the direction of X. To see this, recall that ϕ_t is an integral curve for X passing through p. So

   $$(\phi_{h*} Y)_p = \phi_{h*}(Y_{\phi_{-h}}(p)$$

   moves backward along the flow of X to the point $\phi_{-h}(p)$ (recall that $\phi_{-h} = \phi_h^{-1}$), evaluates Y at that point, and pushes the resulting vector forward to a vector in $T_p M$ via ϕ_{h*}.

   c. If ω is a form on M, then the Lie derivative of ω with respect to X is defined by

   $$L_X \omega(p) := \lim_{h \to 0} \frac{1}{h} \left[\omega_p - (\phi_h^* \omega)_p \right].$$

9. For two vector fields X and Y, define a vector field XY by

   $$XYf = X(Yf).$$

   In coordinates, if

   $$X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i},$$

   and

   $$Y = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i},$$

   then

   $$XYf = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} \left(\sum_{j=1}^{n} b^j \frac{\partial}{\partial x^j} \right) = \sum_{i,j} a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^i} + a^i b^j \frac{\partial^2 f}{\partial x^j \partial x^i}.$$

   Never use this formula for explicit computations. It will drive you insane. Instead, think of X as a differential operator acting on the coefficients of Y.

10. In coordinates, with X and Y as above,

$$L_X Y = XY - YX = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

This formula is also very difficult to use. It's much easier to just compute XY and YX separately as described above.

11. $$L_X dx^i = \sum_{j=1}^{n} \frac{\partial a^i}{\partial x^j} dx^j.$$

12. Properties of Lie derivatives:

    a. $L_X(Y_1 + Y_2) = L_X Y_1 + L_X Y_2$
    b. $L_X(\omega_1 + \omega_2) = L_X \omega_1 + L_X \omega_2$
    c. $L_X fY = Xf \cdot Y + f L_X Y$
    d. $L_X f\omega = Xf \cdot \omega + f L_X \omega$

13. **Cartan's Magic Formula:** If X is a vector field and ω is a form on M, then

$$L_X \omega = d(i_X \omega) + i_X(d\omega).$$

This formula can make the computation of Lie derivatives much simpler. For instance, if ω is closed, the formula reduces to

$$L_X \omega = d(i_X \omega).$$

14. Change of Coordinates: If $\{x^1, \ldots, x^n\}$ and $\{y^1, \ldots, y^n\}$ are coordinate charts on an open subset U of M, and

$$V = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}$$

is a vector field on U written in the x coordinate, then in the y coordinate,

$$V = \sum_i \sum_j x^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}.$$

Integration

1. **Outline of Integration:** The following is summarized from (Spivak Chapter 8), with some added explanations.

   a. A *singular k-cube* is a continuous map from $[0,1]^k$ to M.

   b. You can define the integral of a k-form, ω, over a singular k-cube, c. Denote this by $\int_c \omega$.

   c. For a fixed ω, if c_1 and c_2 are orientation preserving k-cubes which can be extended to diffeomorphisms in a neighborhood of $c[0,1]^k$ such that $Supp(\omega) \subset c_1([0,1]^k)$, $Supp(\omega) \subset c_2([0,1]^k)$, then

   $$\int_{c_1} \omega = \int_{c_2} \omega.$$

   So if you want to integrate a form, you can use any k-cube of this type such that its image contains the support of ω. You will always get the same result. Therefore, for a given ω, we can use

   $$\int_M \omega$$

   to denote the integral of ω over *any* k-cube with the above properties, and this is a well-defined quantity.

   d. Using the fact stated above, we can define integration of a form with compact support on an orientable k-manifold. (The manifold has to be orientable because of the 'orientation preserving' part of the above statement.) The simplest thing to try would be the following: Given a k-form, ω, with compact support, choose a k-cube, c, such that c can be extended to a diffeomorphism on a neighborhood of $[0,1]^k$ and $Supp(\omega) \subset c([0,1]^k$. Now just compute $\int_c \omega$. The problem with this approach is that there may not be any k-cube with these properties. (For example, let $M = S^1$ and

   $$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

   Then $Supp(\omega) = S^1$, and there is no diffeomorphism from $[0,1]$ *onto* S^1.) So we need to break M up into pieces, define integration on the pieces, and glue them together using a partition of unity.

   Here are the details. First, choose an open cover $\mathfrak{U}$ of M such that each $U \in \mathfrak{U}$ is contained in $c[0,1]^k$ where c is an orientation preserving k-cube which can be extended to a diffeomorphism on a neighborhood of $[0,1]^k$. Let Φ be a partition of unity subordinate to $\mathfrak{U}$. Then

   $$\int_M \phi \cdot \omega$$

   is well-defined for each $\phi \in \Phi$. Now we can define

   $$\int_M \omega := \sum_{\phi \in \Phi} \int_M \phi \cdot \omega.$$

   Note that since ω has compact support, this is a finite sum. In fact, if $Supp(\omega)$ is not compact, then issues of convergence could arise, so this definition might not always make sense.

2. **Stokes' Theorem for Manifolds with Boundary:** If M is an orientable k-manifold with boundary, and ω is a $(k-1)$-form with compact support, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

To integrate over M, we need to choose an orientation. This induces an orientation on ∂M, and the integral on the right is with respect to this induced orientation.

Orientable Manifolds

1. **Theorem (Spivak 209-210).** If M is a C^∞ n-manifold, then M is orientable iff there exists an n-form on M which is nowhere zero (i.e., a nonvanishing section of the cotangent bundle).

Differential Equations

The differential equations that show up on the quals have so far all been second order differential equations with constant coefficients. That is, equations of the form

$$y'' + a_1 y' + a_2 y = 0.$$

To find the general solution to this kind of equation, first, write down the corresponding "Auxiliary Equation"

$$r^2 + a_1 r + a_0 = 0.$$

Second, find the solutions, r_1 and r_2 to the auxiliary equations using the quadratic formula. Now you can immediately write down the general solution. There are three cases.

1. r_1 and r_2 are real and distinct. In this case, the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

2. $r_1 = r_2$ is real. In this case, the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}.$$

3. r_1 and r_2 are complex conjugates with $r_1 = a + ib$ and $r_2 = a - ib$. In this case, the general solution is

$$y(x) = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx.$$

Stereographic Projection

There is a diffeomorphism

$$\varphi : S^{n-1} \setminus \{\text{north pole}\} \longrightarrow \mathbb{R}^{n-1}$$

defined by

$$\varphi((x_1, x_2, \ldots, x_n)) = \left(\frac{x_1}{1 - x_n}, \ldots, \frac{x_{n-1}}{1 - x_n} \right).$$

We also have

$$\varphi^{-1} : \mathbb{R}^{n-1} \longrightarrow S^{n-1} \setminus \{\text{north pole}\}$$

defined by

$$\varphi^{-1}((u_1, \ldots, u_{n-1})) = \left(\frac{2u_1}{u_1^2 + \cdots + u_{n-1}^2 + 1}, \ldots, \frac{2u_{n-1}}{u_1^2 + \cdots + u_{n-1}^2 + 1}, \frac{-2}{u_1^2 + \cdots + u_{n-1}^2 + 1} + 1 \right).$$

Questions

a. Does homotopy of topological spaces (i.e., images of maps from S^n for some n) preserve homology groups?

b. How does one compute cohomology groups of cartesian products of spaces?

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