MATH534A Final Exam · Due Thursday December 13

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1. Let S be the unit square. The Möbius band is the quotient of S by the equivalence relation $(0,y) \overset{M}{\sim} (1,1-y)$. The Klein bottle is the quotient of S by the equivalence relation $(0,y) \overset{K}{\sim} (1,1-y)$, $(x,0) \overset{K}{\sim} (x,1)$.
   (a) Draw the square S and indicate the identifications that produce the Möbius band. Then using this picture of $S/\overset{M}{\sim}$, explain why the boundary of the Möbius band is a single circle.

   The boundary of the Möbius band is just the top and bottom of the square (the left and right sides are identified so not on the boundary). Start at $(1/2,1)$ and travel right. You will hit the point $(1,1)$ which is the same as $(0,0)$. Continuing in the same direction you will cover the bottom of the boundary then reach $(1,0)$ which is identified with $(0,1)$. Continuing to travel right you will get back to your starting position as pictured above.
   So the entire boundary can be covered by travelling in one direction and you reach your starting point heading in the same direction. So it must be homeomorphic to a circle.

   (b) We will cut up the Klein bottle $S/\overset{K}{\sim}$. Cut S into five horizontal strips of equal width $x \in [0,1]$. Label these strips A, B, C, D, E bottom to top.

   What topological spaces are:
   (i) $A \cup E$.

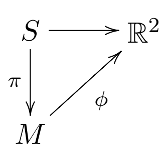
   Placing E below A so that the identified line $(x,0) \overset{K}{\sim} (x,1)$ becomes one line we see that the vertical edges are identified but in opposite directions. Bringing these together $A \cup E$ is a Möbius band.

   (ii) $B \cup D$.

   Placing B and D next to each other with one of their identified edges together we see that the other identified edge is facing in the right direction so can be brought together to form a cylinder.

   (iii) C is clearly a Möbius band.

2. Let M be the set of all lines in $\mathbb{R}^2$.
   (a) Define the structure of a topological manifold on M.
   Recall that every line in $\mathbb{R}^2$ can be described by an equation $ax + by + c = 0$ where not both a and b are zero. Also note that two equations of this form describe the same line iff one is a non-zero multiple of the other.
   So we define an equivalence relation $\sim$ on $S = \mathbb{R}^3 \setminus \{(0, 0, z) \mid z \in \mathbb{R}\}$ by $(a, b, c) \sim (a', b', c')$ if there exists $k \in \mathbb{R} \setminus \{0\}$ such that $(a', b', c') = k(a, b, c)$. Then S inherits a topology as a subspace of $\mathbb{R}^3$.
   We check that $\sim$ is an equivalence relation: it is symmetric because $(a, b, c) = 1(a, b, c)$, it is reflexive because $(a', b', c') = k(a, b, c)$ implies $(a, b, c) = 1/k(a', b', c')$ and transitive because $(a', b', c') = k(a, b, c)$ and $(a'', b'', c'') = l(a', b', c')$ implies $(a'', b'', c'') = kl(a, b, c)$. So we can form $M = S/\sim$ and give it the quotient topology.
   Now we need to show that M is Hausdorff, second countable and locally Euclidean (of dimension two). We will write $[a, b, c]$ for the equivalence class of (a, b, c).
      (i) To see that M is Hausdorff we need to show that for any $[a, b, c], [d, e, f] \in M$ there exists disjoint open sets around them. To do this recall that a subset of M is open iff $\pi^{-1}(U)$ is open where π is the projection map $\pi\colon S \to B$.
      Also recall that $U \subseteq S$ is saturated if $U = \pi^{-1}(\pi(U))$ and that the quotient map π takes saturated open sets to open sets (to see this note that $\pi^{-1}(\pi(U)) = U$ is open so $\pi(U)$ is).
      Now, notice that right double cones through the origin (but not including the origin) are saturated: Let U be a right double cone through the origin. If $(a, b, c) \in \pi^{-1}(\pi(U))$ then $(a, b, c) \sim (a', b', c') \in U$ so is on the same line which is contained in the cone so $(a, b, c) \in U$.
      If $[a, b, c] \neq [d, e, f]$ then $(a, b, c) \not\sim (d, e, f)$ so we can find two non-intersecting open right double cones through the origin containing these two points. Then the image of these two open saturated sets under π are open sets in M containing $[a, b, c]$ and $[d, e, f]$ and not intersecting (because if they did then they would intersect in S). Therefore M is Hausdorff.
      (ii) To see that M is locally Euclidean of dimension 2 we provide charts.
      Let $U_a = \{[a, b, c] \mid a \neq 0\}$ and $\phi_a\colon U_a \to \mathbb{R}^2$ be defined by $\phi_a\colon [a, b, c] \mapsto (b/a, c/a)$.



      We check that ϕ_a is well defined because $\phi_a\colon [ka, kb, kc] \mapsto (kb/ka, kc/ka) = (b/a, c/a)$. Now $\tilde{U}_a = \{(a, b, c) \mid a \neq 0\}$ is a saturated open set so $U_a = \pi(\tilde{U}_a)$ is open. Recall that ϕ_a is continuous iff $\phi_a \circ \pi$ is. But $\phi_a \circ \pi\colon \tilde{U}_a \to \mathbb{R}^2$ is given by $\phi_a \circ \pi\colon (a, b, c) \mapsto (b/a, c/a)$ so is clearly continuous. Now $\phi_a(U_a) = \{(b/a, c/a) \mid a, b, c \in \mathbb{R}, a \neq 0\} = \mathbb{R}^2$ is open in $\mathbb{R}^2$. The map $\phi_a^{-1}\colon (u, v) \mapsto [1, u, v]$ is continuous so ϕ_a is a homeomorphism so U_a is homeomorphic to an open subset of $\mathbb{R}^2$.
      Similarly, define $U_b = \{[a, b, c] \mid b \neq 0\}$ and $\phi_b\colon U_b \to \mathbb{R}^2$ by $\phi_b\colon [a, b, c] \mapsto (a/b, c/b)$. By exactly the same argument U_b is an open set of M homeomorphic to $\phi_b(U_b)$, an open set of $\mathbb{R}^2$.
      So since U_a, U_b cover M we see that M is locally Euclidean.
      (iii) To see that M is second countable we need to show that there exists a countable basis for the topology.
      But U_a, U_b cover M and each is homeomorphic to a subset of $\mathbb{R}^2$ which is second countable so U_a has a countable basis and U_b has a countable basis. The union of these two sets is a countable basis for M.

(b) Define the structure of a smooth manifold on M.
We show that $\{(U_a, \phi_a), (U_b, \phi_b)\}$ is a smooth atlas on M. For this we need $\phi_b^{-1} \circ \phi_a \colon \phi_b(U_a \cap U_b) \to \phi_a(U_a \cap U_b)$ smooth and similarly $\phi_a^{-1} \circ \phi_b$ smooth.
But $\phi_b^{-1} \circ \phi_a \colon (u, v) \mapsto (1/u, v/u)$ which is clearly smooth (has partial derivatives of all orders) because $u \neq 0$. Similarly $\phi_a^{-1} \circ \phi_b \colon (u, v) \mapsto (1/u, v/u)$ is smooth.

(c) Show that M is homeomorphic to the Möbius band of infinite width given by $\{(x, y) \mid x \in [0, 1], y \in \mathbb{R}\}$ modulo the equivalence relation $(0, y) \sim (1, -y)$.
We can take a representative of each equivalence class in S to lie on the infinite vertical cylinder of radius 1. On this cylinder we have $(a, b, c) \sim (-a, -b, -c)$ and $\sqrt{a^2 + b^2} = 1$. If we now consider only the half of the cylinder with $a \geq 0$ then this is an infinitely long curved strip with the left and right edges identified in opposite directions. Projecting onto the y-z plane we get the infinite Möbius band $N = \{(y, z) \mid y \in [-1, 1], z \in \mathbb{R}\}$ modulo $(-1, z) \sim (1, -z)$, which is clearly homeomorphic to the infinite Möbius band described in the question (just shift one unit in the positive y direction and then halve the y–coordinate).
The map $\phi \colon M \to N$ is given by

$$\phi \colon [a, b, c] \mapsto \operatorname{sgn}(a) \left(\frac{b}{\sqrt{a^2 + b^2}}, \frac{c}{\sqrt{a^2 + b^2}} \right).$$

To see ϕ is well defined we check, for all $k \neq 0$

$$\begin{aligned} \phi[ka, kb, kc] &= \operatorname{sgn}(ka) \left(\frac{kb}{\sqrt{(ka)^2 + (kb)^2}}, \frac{kc}{\sqrt{(ka)^2 + (kb)^2}} \right) \\ &= \operatorname{sgn}(k) \operatorname{sgn}(a) \left(\frac{kb}{|k|\sqrt{a^2 + b^2}}, \frac{kc}{|k|\sqrt{a^2 + b^2}} \right) \\ &= \operatorname{sgn}(a) \left(\frac{b}{\sqrt{a^2 + b^2}}, \frac{c}{\sqrt{a^2 + b^2}} \right) = \phi[a, b, c]. \end{aligned}$$

The inverse of ϕ is given by

$$\phi^{-1}(y, z) = [\sqrt{1 - y^2}, y, z].$$

At $(1, z) \sim (-1, -z)$ we check that ϕ^{-1} is well defined

$$\phi^{-1}(1, z) = [0, 1, z] = [0, -1, -z] = \phi^{-1}(-1, -z).$$

We check

$$\phi(\phi^{-1}(y, z)) = \phi[\sqrt{1 - y^2}, y, z] = \left(\frac{1 \cdot y}{1 - y^2 + y^2}, \frac{1 \cdot z}{1 - y^2 + y^2} \right) = (y, z)$$

and

$$\begin{aligned} \phi^{-1}(\phi[a, b, c]) &= \phi^{-1} \left(\operatorname{sgn}(a) \left(\frac{b}{\sqrt{a^2 + b^2}}, \frac{c}{\sqrt{a^2 + b^2}} \right) \right) \\ &= \left[\sqrt{1 - \left(\frac{\operatorname{sgn}(a) b}{\sqrt{a^2 + b^2}} \right)^2}, \frac{\operatorname{sgn}(a) b}{\sqrt{a^2 + b^2}}, \frac{\operatorname{sgn}(a) c}{\sqrt{a^2 + b^2}} \right] \\ &= \left[\frac{\sqrt{a^2 + b^2 - b^2}}{\sqrt{a^2 + b^2}}, \frac{\operatorname{sgn}(a) b}{\sqrt{a^2 + b^2}}, \frac{\operatorname{sgn}(a) c}{\sqrt{a^2 + b^2}} \right] \\ &= \left[\frac{\operatorname{sgn}(a) a}{\sqrt{a^2 + b^2}}, \frac{\operatorname{sgn}(a) b}{\sqrt{a^2 + b^2}}, \frac{\operatorname{sgn}(a) c}{\sqrt{a^2 + b^2}} \right] = [a, b, c]. \end{aligned}$$

Thus ϕ is a bijection and since ϕ and ϕ^{-1} are clearly continuous we have that M and N are homeomorphic, that is, the set of lines in $\mathbb{R}^2$ is homeomorphic to the infinite Möbius band.

3. Prove that there exists a coordinate system (s,t) in a neighbourhood of the point $x=y=0$ in which the vector fields

$$\boldsymbol{X}=(1+y)\frac{\partial}{\partial x}+x\frac{\partial}{\partial y} \text{ and } \boldsymbol{Y}=x\frac{\partial}{\partial x}+(1+y)\frac{\partial}{\partial y}$$

take the form $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ respectively. Find this coordinate system.
First we check that the Lie bracket of these two vector fields is zero:

$$[\boldsymbol{X},\boldsymbol{Y}]=((1+y)(1)+0-0-(1+y)(1))\frac{\partial}{\partial x}+(0+(x)(1)-(x)(1)-0)\frac{\partial}{\partial y}=0.$$

This mean, by the argument in the Lie Brackets handout, that the integral curves form a coordinate grid with mixed partials equal. Then in a neighbourhood around the origin we can take some combination of these two coordinates as our desired coordinates.

To find the coordinate system we pullback the vector field on the identity (just using the chain rule really) and find

$$\begin{aligned}\boldsymbol{X} &= (1+y)\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\\ &= (1+y)\left(\frac{\partial s}{\partial x}\frac{\partial}{\partial s}+\frac{\partial t}{\partial x}\frac{\partial}{\partial t}\right)+x\left(\frac{\partial s}{\partial y}\frac{\partial}{\partial s}+\frac{\partial t}{\partial y}\frac{\partial}{\partial t}\right)\\ &= \left((1+y)\frac{\partial s}{\partial x}+x\frac{\partial s}{\partial y}\right)\frac{\partial}{\partial s}+\left((1+y)\frac{\partial t}{\partial x}+x\frac{\partial t}{\partial y}\right)\frac{\partial}{\partial t}\end{aligned}$$

and similarly

$$\boldsymbol{Y}=\left(x\frac{\partial s}{\partial x}+(1+y)\frac{\partial s}{\partial y}\right)\frac{\partial}{\partial s}+\left(x\frac{\partial t}{\partial x}+(1+y)\frac{\partial t}{\partial y}\right)\frac{\partial}{\partial t}.$$

So for $\boldsymbol{X}=\frac{\partial}{\partial s}$ and $\boldsymbol{Y}=\frac{\partial}{\partial t}$ we must have

$$(1+y)\frac{\partial s}{\partial x}+x\frac{\partial s}{\partial y}=1 \tag{1}$$

$$(1+y)\frac{\partial t}{\partial x}+x\frac{\partial t}{\partial y}=0 \tag{2}$$

$$x\frac{\partial s}{\partial x}+(1+y)\frac{\partial s}{\partial y}=0 \tag{3}$$

$$x\frac{\partial t}{\partial x}+(1+y)\frac{\partial t}{\partial y}=1. \tag{4}$$

Now, $(1)-\frac{1+y}{x}(3)$ gives the equation

$$x\frac{\partial s}{\partial y}-\frac{(1+y)^2}{x}\frac{\partial s}{\partial y}=1$$

which tells us that

$$\frac{\partial s}{\partial y}=\frac{x}{x^2-(1+y)^2}.$$

Then

$$\frac{\partial s}{\partial x}=-\frac{1+y}{x}\frac{\partial s}{\partial y}=\frac{1+y}{(1+y)^2-x^2}.$$

So

$$\begin{aligned} s(x,y) &= \int \frac{1+y}{(1+y)^2 - x^2}\,dx \\ &= \int \frac{1}{2}\left(\frac{1}{1+y+x} + \frac{1}{1+y-x}\right)dx \\ &= \frac{1}{2}(\ln(1+y+x) - \ln(1+y-x)) + f(y) \\ &= \frac{1}{2}\ln\left(\frac{1+y+x}{1+y-x}\right) + f(y). \end{aligned}$$

where we are using the fact that we are in a sufficiently small neighbourhood of the origin to say $1+y+x$ and $1+y-x$ are positive.

Then

$$\begin{aligned} \frac{\partial s}{\partial y} &= \frac{1}{2}\left(\frac{(1+y+x)\cdot 1 - (1+y-x)\cdot 1}{(1+y-x)^2}\right) / \left(\frac{1+y+x}{1+y-x}\right) + f'(y) \\ &= \frac{1}{2}\frac{-2x}{(1+y-x)(1+y+x)} + f'(y) \\ &= \frac{x}{x^2-(1+y)^2} + f'(y) \end{aligned}$$

So then $f'(y) = 0$ and f is constant. We take $f(y) = 0$ so that $x = y = 0$ corresponds with $s = 0$.

Fairly similarly, $(4) - \frac{x}{1+y}(2)$ gives us

$$\frac{\partial t}{\partial y} = \frac{1+y}{(1+y)^2 - x^2} \text{ and then } \frac{\partial t}{\partial x} = \frac{x}{x^2-(1+y)^2}.$$

So

$$\begin{aligned} t(x,y) &= \int \frac{x}{x^2-(1+y)^2}\,dx \\ &= \frac{1}{2}\ln(x^2-(1+y)^2) + g(y). \end{aligned}$$

where we can check again that we can take $g(y) = 0$.

So the coordinate system is

$$s = \frac{1}{2}\ln\left(\frac{1+y+x}{1+y-x}\right) \text{ and } t = \frac{1}{2}\ln(x^2-(1+y)^2).$$

4. The manifold is $\mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n)$. Let $f(\boldsymbol{x}) = \sum_{j=1}^n (x^j)^{a_j}$, where the a_j are positive integers. Find an $(n-1)$-form η such that

$$df \wedge \eta = f(\boldsymbol{x})\, dx^1 \wedge \cdots \wedge dx^n.$$

Take

$$\eta = g_j(\boldsymbol{x})\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$

where the hat indicates that dx^j is not included in the jth summand.

Then

$$df = \frac{\partial f}{\partial x^i} dx^i = a_i (x^i)^{a_i - 1} dx^i$$

so

$$\begin{aligned} df \wedge \eta &= \sum_{i=1}^n \left(a_i (x^i)^{a_i-1} g_i(\boldsymbol{x})\, dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n \right) \\ &= \left(\sum_{i=1}^n a_i (x^i)^{a_i-1} g_i(\boldsymbol{x}) \right) dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Now if we take $g_i(\boldsymbol{x}) = x_i / a_i$ then

$$\sum_{i=1}^n a_i (x^i)^{a_i-1} g_i(\boldsymbol{x}) = \sum_{i=1}^n (x_i^{a_i}) = f(\boldsymbol{x})$$

so $df \wedge \eta = f(\boldsymbol{x})\, dx^1 \wedge \cdots \wedge dx^n$ as required.

5. Let V be an n–dimensional vector space and $\Lambda^k(V)$ be the vector space of alternating tensors of degree $k \leq n$. For $\eta \in \Lambda^k(V)$, say $v \in V$ *divides* η if there exists $\mu \in \Lambda^{k-1}(V)$ such that $\eta = \mu \wedge v$. Prove that v divides η iff $\eta \wedge v = 0$.

   If $v \in V$ divides $\eta \in \Lambda^k(V)$ there exists $\mu \in \Lambda^{k-1}(V)$ such that $\eta = \mu \wedge v$. So then
$$\eta \wedge v = (\mu \wedge v) \wedge v = \mu \wedge (v \wedge v) = \mu \wedge 0 = 0.$$

   Conversely we will work with a basis $\{E_1, \ldots, E_n\}$ of V with $E_n = v$ (recall from linear algebra that any linearly independent set can be extended to a basis so this can be done).

   Then if we have a multi–index I (a list of k of the numbers $1, \ldots, n$ in ascending order) write $E_I = E_{i_1} \wedge \cdots \wedge E_{i_n}$ and then $\{E_I\}$ is a basis for $\Lambda^k(V)$.

   Write
$$\eta = \sum_{|I|=k} \eta_I E_I$$
and then
$$\eta \wedge v = \sum_{|I|=k, n \notin I} \eta_I E_I \wedge E_n.$$
But by assumption this is zero, so $\eta_I = 0$ for all I with $n \notin I$. Therefore
$$\eta = \sum_{|I|=k, n \in I} \eta_I E_I = \left(\sum_{|I|=k, n \in I} \eta_I E_{I \setminus \{n\}} \right) \wedge E_n.$$
So if we let
$$\mu = \sum_{|J|=k-1, n \notin J} \eta_{J \cup \{n\}} E_J$$
then $\eta = \mu \wedge v$ and v divides η as desired.

6. Let $M = S^2$ be the standard 2–sphere in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 = 1$. Define two vector fields on $\mathbb{R}^3$ by

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \text{ and } Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}.$$

(a) Show that X is tangent to M and so defines a vector field, called X_M on M. Likewise for Y, producing a vector field Y_M on M.
At $p = (x_0, y_0, z_0) \in M$ we know that the normal vector to M is $N = x_0\frac{\partial}{\partial x} + y_0\frac{\partial}{\partial y} + z_0\frac{\partial}{\partial z}$. Then $X_p = 0\frac{\partial}{\partial x} + z_0\frac{\partial}{\partial y} - y_0\frac{\partial}{\partial z}$ so $\langle N, X_p\rangle = y_0 z_0 - z_0 y_0 = 0$. Therefore X_p is orthogonal to the normal vector so is tangent to the sphere M.
Then, identifying the tangent space to M at p as a subspace of the tangent space to $\mathbb{R}^3$ at p in the obvious way, we see that $X_p \in T_p(M)$. So X_M can be defined simply as the restriction of X to M.
Similarly $\langle N, Y_p\rangle = -x_0 z_0 + z_0 x_0 = 0$ so Y is tangent to M and defining Y_M to be the restriction of Y to M we see that Y_M is a vector field on M.

(b) Compute $[X_M, Y_M]$.
First we calculate $[X, Y]$ in $\mathbb{R}^3$. If $X = X^i\frac{\partial}{\partial x^i}$ and $Y = y^j\frac{\partial}{\partial x^j}$ then, by Lemma 4.13,

$$\begin{aligned}
[X,Y] &= \left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j} \\
&= (0+0+(-y)(-1)-0-0-0)\frac{\partial}{\partial x} + (0+0+0-0-0-(x)(1))\frac{\partial}{\partial y} + (0+0+0-0-0-0)\frac{\partial}{\partial z} \\
&= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.
\end{aligned}$$

Notice that this is tangent to M so restricts to a vector field $[X,Y]_M$ on M. Now the vector fields X_M and X are the same at a point $p \in M$, that is $(X_M)_p = X_p$and similarly for Y. So for all $f \in C^\infty(M)$,

$$\begin{aligned}
[X_M, Y_M]_p f &= (X_M)_p(Y_M)_p f - (Y_M)_p(X_M)_p f \\
&= X_p Y_p f - Y_p X_p f \\
&= [X,Y]_p f.
\end{aligned}$$

So $[X_M, Y_M]_p = [X,Y]_p$ for all $p \in M$, that is $[X_M, Y_M] = [X,Y]_M$, which is the restriction of the vector field calculated above

(c) Let C be the circle defined by $y^2 + z^2 = 1$ and $x = 0$. Let $C_t = F_t(C)$ where F_t is the flow of X. Describe C_t geometrically.
Recall $F_t(x_0, y_0, z_0) = \gamma(t) = (x(t), y(t), z(t))$ where $\gamma'(s) = X_{\gamma(s)}$ for all $s \in [0,t]$ and $\gamma(0) = (x_0, y_0, z_0)$.
So if $(0, y_0, z_0) \in C$ we require

$$\frac{\partial x}{\partial t} = 0,\ \frac{\partial y}{\partial t} = z \text{ and } \frac{\partial z}{\partial t} = -y$$

with the initial conditions $x(0) = 0, y(0) = y_0$ and $z(0) = z_0$.
These equations have the solution

$$F_t(0, y_0, z_0) = (x(t), y(t), z(t)) = (0, y_0\cos t + z_0 \sin t, -y_0 \sin t + z_0\cos t).$$

Geometrically F_t is a rotation of t radians anticlockwise in the y–z plane. So $C_t = F_t(C) = C$ because rotating a circle in the plane it lies in has no effect.

(d) Let α be the one–form on $\mathbb{R}^3$ defined by

$$\alpha = yz\,dx + xz\,dy + xy\,dz$$

and α_M be the pullback of α from $\mathbb{R}^3$ to M. Compute $d\alpha_M$.

If $i\colon M \to \mathbb{R}^3$ is the inclusion map then $\alpha_M = i^*(\alpha)$. Then $d\alpha_M = d(i^*\alpha) = i^*(d\alpha)$ because the pullback map commutes with d by Lemma 12.16. Now

$$\begin{aligned} d\alpha &= (y\,dz + z\,dy) \wedge dx + (x\,dz + z\,dx) \wedge dy + (x\,dy + y\,dx) \wedge dz \\ &= (z - z)dx \wedge dy + (x - x)dy \wedge dz + (y - y)dx \wedge dz = 0 \end{aligned}$$

Therefore $d\alpha_M = i^*(0) = 0$.

(e) Compute

$$\int_C \alpha_M.$$

The circle C can be parametrised by $\gamma\colon [0, 2\pi] \to M$ with $\gamma(t) = (0, \cos t, \sin t)$. Then

$$\begin{aligned} \int_C \alpha_M &= \int_{[0,2\pi]} \gamma^* \alpha_M \\ &= \int_0^{2\pi} \cos t \sin t\, d(0) + 0 \sin t\, d(\cos t) + 0 \cos t\, d(\sin t) \\ &= \int_0^{2\pi} 0\, dt = 0. \end{aligned}$$

7. Let M be a manifold (possibly nonorientable). Prove that the tangent bundle TM is an orientable manifold.

We provide the standard charts for TM and check that they are consistently oriented. Then Proposition 13.3 tells us that the manifold TM is orientable.

If $\{(U_\alpha, \phi_\alpha)\}$ is a smooth atlas for M and $\pi\colon TM \to M$ is the projection map then we will use charts $(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)$ where $\tilde{\phi}\colon \pi^{-1}(U) \to \mathbb{R}^{2n}$ is given by

$$\tilde{\phi}\left(v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = (x^1(p), \ldots, x^n(p), v^1, \ldots, v^n).$$

Note that

$$\tilde{\phi}^{-1}(x^1, \ldots, x^n, v^1, \ldots, v^n) = v^i \left.\frac{\partial}{\partial x^i}\right|_{\phi^{-1}(x)}.$$

Now if (U_α, ϕ_α) has coordinates x^i and (U_β, ϕ_β) has coordinates y_i then $\phi_\beta \circ \phi_\alpha^{-1}(x) = (y^1(x), \ldots, y^n(x))$.

So, $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is given by

$$\begin{aligned} & (\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1})(x^1, \ldots, x^n, v^1, \ldots, v^n) \\ = \ & \tilde{\phi}_\beta\left(v^i \left.\frac{\partial}{\partial x^i}\right|_{\phi_\alpha^{-1}(x)}\right) \\ = \ & \tilde{\phi}_\beta\left(v^j \frac{\partial y^i}{\partial x_j}(x) \left.\frac{\partial}{\partial y^i}\right|_{\phi_\alpha^{-1}(x)}\right) \\ = \ & \left(y^1(x), \ldots, y^n(x), v^j \frac{\partial y^1}{\partial x^j}, \ldots, v^j \frac{\partial y^n}{\partial x^j}\right). \end{aligned}$$

These charts will be consistently oriented if the Jacobian determinant of $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ is positive.

Now we consider the derivatives in this Jacobian. First note that $\frac{\partial y^i}{\partial v^j} = 0$ for all i, j because y has no v dependence. Thus the upper right quarter of the matrix is zero so the determinant is the multiple of the determinant of the upper left quarter and the determinant of the bottom right quarter.

The upper left quarter has (i, j) entry equal to $\frac{\partial y^i}{\partial x^j}$. The bottom right quarter has (i, j) term

$$\frac{\partial}{\partial v^j} v^k \frac{\partial y^i}{\partial x^k} = \frac{\partial y^i}{\partial x^j}.$$

Thus the Jacobian determinant is the square of the Jacobian determinant of the coordinate functions y, which is non-zero. Therefore the Jacobian determinant is positive and the manifold TM is orientable.