MATH523B Final Exam · Due Wednesday May 14

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(1) Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measure spaces and $\phi\colon X \to Y$ be a measurable function. For a measure μ on $(X, \mathfrak{M})$ define a measure $\phi_*\mu$ on $(Y, \mathfrak{N})$ by

$$(\phi_*\mu)(A) = \mu(\phi^{-1}(A)), \qquad A \in \mathfrak{N}.$$

Let μ and ν be measures on $(X, \mathfrak{M})$.

(a) Prove that $\mu \ll \nu$ implies $\phi_*\mu \ll \phi_*\nu$.

(b) Does $\phi_*\mu \ll \phi_*\nu$ imply $\mu \ll \nu$?

(a) If $(\phi_*\nu)(A) = 0$ then $\nu(\phi^{-1}(A)) = 0$. But then $\mu(\phi^{-1}(A)) = 0$ because $\mu \ll \nu$. So $(\phi_*\mu)(A) = 0$ and since A was arbitrary, $\phi_*\mu \ll \phi_*\nu$.

(b) No. To see this take μ not absolutely continuous with respect to ν and define $\phi\colon X \to Y$ by $\phi(x) = a$ for some fixed $a \in Y$. Then we see for $A \subseteq Y$, $\phi^{-1}(A) = X$ if $a \in Y$ and is empty otherwise so ϕ is measurable. Then if $(\phi_*\nu)(A) = \nu(\phi^{-1}(A)) = 0$ we see (assuming ν nontrivial) that $\phi^{-1}(A) = \emptyset$. Thus $(\phi_*\mu)(A) = \mu(\phi^{-1}(A)) = 0$ also and $\phi_*\mu \ll \phi_*\nu$.

(2) Let H be an infinite dimensional Hilbert space. Prove that it is impossible to construct a measure on H such that all balls $B(x, r)$ with $r > 0$ are measurable with the measure of $B(x, r)$ finite, positive and depending on the radius r only.

Say μ is a measure as described in the question. Since H is infinite dimensional we can find a countable orthonormal set $\{u_i\}$ indexed by the natural numbers. Now each $u_i \in B(0, 2)$ and we can form $B(u_i, 1/4)$ which is contained in $B(0, 2)$ because if $x \in B(u_i, 1/4)$ then $||x|| \leq ||x - u_i|| + ||u_i|| \leq 5/4 < 2$.

But these balls of radius $1/4$ are disjoint: if $x \in B(u_i, 1/4) \cap B(u_j, 1/4)$ then

$$||u_i - u_j|| \leq ||u_i - x|| + ||x - u_j|| \leq 1/2.$$

But $||u_i - u_j|| = \sqrt{(u_i - u_j, u_i - u_j)} = \sqrt{2}$ if $i \neq j$ so the intersection is empty.

Thus $B(u_i, 1/4)$ is a sequence of disjoint measurable sets all contained in $B(0, 2)$ so

$$\sum_{i=1}^{\infty} \mu(B(u_i, 1/4)) \leq \mu(B(0, 2)).$$

By assumption all of the terms in the left hand sum are equal positive numbers and the right hand side is a finite number. This is a contradiction and such a measure cannot exist.

(3) Let $x = 0.a_1a_2\dots$ be the decimal expansion of a number x with $0 < x < 1$ (if two decimal expansions exist take the one ending with 0s). For what values of $q > 1$ is the function

$$f_q(x) = \sum_{k=1}^{\infty} q^{-k} a_k$$

of bounded variation?

For f_q to be of bounded variation we need

$$T(1) - T(0) = \sup_{\{0<x_1<\cdots<x_n<1\}} \sum_{i=1}^{n-1} |f_q(x_{i+1}) - f_q(x_i)| < \infty.$$

If $q \geq 10$ then we show that f_q is nondecreasing: If $x = \sum_{k=1}^{\infty} a_k 10^{-k} < y = \sum_{k=1}^{\infty} b_k 10^{-k}$ then there exists an $N \geq 1$ such that $a_i = b_i$ for all $i < N$ and $a_N < b_N$. Then

$$f_q(y) - f_q(x) = (b_N - a_N)q^{-N} + \sum_{k=N+1}^{\infty} (b_k - a_k)q^{-k}.$$

But

$$\left| \sum_{k=N+1}^{\infty} (b_k - a_k)q^{-k} \right| \leq \sum_{k=N+1}^{\infty} |b_k - a_k| q^{-k} \leq 9\frac{q^{-(N+1)}}{1-1/q} = \frac{9}{q-1} q^{-N} \leq q^{-N}$$

where the last equality is because $q \geq 10$ so $q - 1 \geq 9$ and $9/(q-1) \leq 1$.

Since $b_N - a_N$ is a positive integer we can conclude that $(b_N - a_N)q^{-N} \geq q^{-N}$ so adding a number smaller or equal in absolute value than this cannot make the answer negative. That is, $f_q(y) - f_q(x) \geq 0$.

So for $q \geq 10$ we see that for any $0 < x_1 < \dots < x_n < 1$

$$\sum_{i=1}^{n-1} |f_q(x_{i+1}) - f_q(x_i)| = \sum_{i=1}^{n-1} (f_q(x_{i+1}) - f_q(x_i)) = f_q(x_n) - f_q(x_1).$$

But

$$f_q(x) = \sum_{k=1}^{\infty} a_k q^{-k} \leq 9\frac{1/q}{1-1/q} = \frac{9}{q-1} \leq 1$$

so each such sum is bounded by 1 and we see that $T(1) - T(0) \leq 1 < \infty$ so f_q is of bounded variation.

For $q < 10$ we show that f_q is not of bounded variation by describing a family of partitions $0 < x_1 < \dots < x_{10^d} < 1$ such that $\sum_{i=1}^{10^d-1} |f_q(x_{i+1}) - f_q(x_i)| \to \infty$ as $d \to \infty$.

To do this write each integer $1 \leq i \leq 10^d - 1$ in decimal notation as $i = b_1 \dots b_d$. For example if $d = 3$ we would write 67 as $b_1 = 0, b_2 = 6, b_3 = 7$. Also find a rational number p such that $q < p < 10$ and p has a finite decimal representation $p = c_1.c_2\dots c_m$ (to do this we can find the first decimal place where q and 10 disagree and then round up to find p).

Then define

$$x_i = 0.b_1 \dots b_d \text{ for } i \text{ odd and}$$

$$x_i = 0.b_1 \dots b_d c_1 \dots c_m c_1 \dots c_m c_1 \dots \text{ for } i \text{ even.}$$

So if i is an odd number whose last digit is not nine then $x_i = 0.b_1 \dots b_{d-1} b_d$ and $x_{i+1} = 0.b_1 \dots b_{d-1}(b_d + 1)c_1 \dots c_m c_1 \dots$.

Therefore

$$\begin{aligned}
f_q(x_{i+1}) - f_q(x_i) &= q^{-d} + q^{-(d+1)} \sum_{l=0}^{\infty} q^{-ml}(c_1 + c_2 q^{-1} + \cdots c_m q^{-m+1}) \\
&\geq q^{-d} + q^{-(d+1)} \sum_{l=0}^{\infty} q^{-ml}(c_1 + c_2 10^{-1} + \cdots c_m 10^{-m+1}) \\
&= q^{-d} + p q^{-(d+1)} \sum_{l=0}^{\infty} q^{-ml} \\
&= q^{-d} + p q^{-(d+1)} \frac{1}{1 - q^{-m}} \\
&> q^{-d} + q q^{-(d+1)} \frac{1}{1 - q^{-m}} \\
&= q^{-d} + \frac{q^{-d} q^m}{q^m - 1} \\
&= C q^{-d}.
\end{aligned}$$

Here C is a positive constant only depending on q and m. Now there are $4 \cdot 10^{d-1}$ such numbers (i odd, last digit not 9 and between 1 and $10^d - 1$). Therefore

$$\sum_{i=1}^{10^d - 1} |f_q(x_{i+1}) - f_q(x_i)| > \frac{4 \cdot 10^d}{10} \frac{C}{q^d} = C' \left(\frac{10}{q} \right)^d \to \infty$$

as $d \to \infty$ because $q < 10$.

(4) Prove that an operator A defined by the formula

$$(Af)(x) = \int_{-\infty}^{\infty} \frac{f(y)}{1 + (x - y)^2} dy$$

is a bounded operator from $L^2(\mathbb{R})$ into itself, compute its norm and prove that A is not invertible.

First we note that if we take $g(x) = 1/(1 + x^2)$ then $g \in L^1(\mathbb{R})$ because

$$\begin{aligned}
||g||_1 &= \int_{-\infty}^{\infty} \frac{1}{1 + x^2} dx \\
&= \lim_{T \to \infty, S \to -\infty} \left(\tan^{-1}(T) - \tan^{-1}(S) \right) \\
&= \pi \\
&< \infty.
\end{aligned}$$

Also, for later use

$$\begin{aligned}
||g||_2^2 &= \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1-x^2}{(1+x^2)^2} + \frac{1+x^2}{(1+x^2)^2} dx \\
&= \frac{1}{2} \left[\frac{x}{1+x^2} + \tan^{-1}(x) \right]_{-\infty}^{\infty} \\
&= \frac{1}{2} \lim_{T\to\infty, S\to -\infty} \left(\frac{T}{1+T^2} + \tan^{-1}(T) - \frac{S}{1+S^2} - \tan^{-1}(S) \right) \\
&= \frac{1}{2}\pi
\end{aligned}$$

so $||g||_2 = \sqrt{\pi/2}$.

Next notice that

$$(g * f)(x) = \int g(x-y)f(y)dy = \int_{-\infty}^{\infty} \frac{f(y)}{1+(x-y)^2} dy = (Af)(x)$$

so $Af = g * f$.

Then we can use Young's inequality to see that $||Af||_2 = ||g * f||_2 \le ||g||_1 \, ||f||_2 = \pi \, ||f||_2 < \infty$ because $f \in L^2(\mathbb{R})$. So A is bounded with $||A||_{L^2 \to L^2} \le \pi$.

If we wanted to show that $||A|| = \pi$ we need either an $f \in L^2(\mathbb{R})$ such that $||Af|| \,/\, ||f|| = \pi$ or a sequence $\{f_n\} \subseteq L^2(\mathbb{R})$ such that $||Af_n|| \,/\, ||f_n|| \to \pi$. I don't see how to do this.

Now if we define $g_n(x) = n \cdot g(nx)$ we see that after the change of variable $z = nx$

$$\begin{aligned}
||g_n||_1 &= \int_{-\infty}^{\infty} \frac{n}{1+(nx)^2} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{1+(z)^2} dz \\
&= \pi
\end{aligned}$$

and

$$\begin{aligned}
||g_n||_2^2 &= \int_{-\infty}^{\infty} \frac{n^2}{(1+(nx)^2)^2} dx \\
&= n \int_{-\infty}^{\infty} \frac{1}{(1+(z)^2)^2} dz \\
&= n\pi/2
\end{aligned}$$

so $||g_n||_2 = \sqrt{n\pi/2}$.

Now by Theorem 8.14a with $\phi = g, g_n = \phi_{1/n}$ and $f = g$ we see that $g * g_n \to \pi g$ in L^2 as $n \to \infty$. That is $||g * g_n||_2 \to \pi \, ||g||_2$ as $n \to \infty$.

Assume that A is a bijection so that A^{-1} exists as a function. We will show that it must be unbounded, that is there exists $f_n \in L^2(\mathbb{R})$ such that $\left|\left|A^{-1} f_n\right|\right|_2 / \, ||f_n||_2 \to \infty$. Take $f_n = Ag_n$, then $\left|\left|A^{-1} f_n\right|\right|_2 / \, ||f_n||_2 = ||g_n||_2 \,/\, ||g * g_n||_2 \to \infty$ because $||g_n||_2 \to \infty$ and $||g * g_n||_2$ converges to a constant as $n \to \infty$.

(5) Let $\Omega \subseteq \mathbb{R}^n$ be a set of nonzero finite Lebesgue measure. Let $\{\psi_j\}$ be an orthonormal basis in $L^2(\Omega)$. Each function ψ_j is extended to $\mathbb{R}^n$ by setting $\psi_j(x) = 0$ when $x \notin \Omega$. Prove that

$$\sum_{j=1}^{\infty} \left|\hat{\psi}_j(\xi)\right|^2 = m(\Omega)$$

for every value of ξ.

We have $m(\Omega) < \infty$ so by proposition 6.12 we see that $L^2(\Omega) \subset L^1(\Omega)$ and $||f||_1 \leq ||f||_2\, m(\Omega)^{1/2}$. So for $f \in L^2(\Omega)$ we can consider it as $f \in L^1(\mathbb{R}^n)$ and then $\hat{f}$ is bounded and continuous with $||\hat{f}||_u \leq ||f||_1$.

Also $\mathcal{F}$ is an unitary isomorphism from $L^2(\mathbb{R}^n)$ to itself so it restricts to a unitary isomorphism from $L^2(\Omega)$ to $\mathcal{F}(L^2(\Omega))$.

Fix $\xi \in \Omega$ and define a function $l \colon \mathcal{F}(L^2(\Omega)) \to \mathbb{C}$ by $l(f) = f(\xi)$. This is linear because $l(cf) = (cf)(\xi) = cf(\xi) = cl(f)$ and $l(f_1 + f_2) = (f_1 + f_2)(\xi) = f_1(\xi) + f_2(\xi) = l(f_1) + l(f_2)$. It is bounded because $|l(\hat{f})| = |\hat{f}(\xi)| \leq ||\hat{f}||_u \leq ||f||_1 \leq ||f||_2\, m(\Omega)^{1/2} = ||\hat{f}||_2 m(\Omega)^{1/2}$ so $||l|| \leq m(\Omega)^{1/2}$.

Therefore $l \in \mathcal{F}(L^2(\Omega))^*$ so there exists a $\theta \in \mathcal{F}(L^2(\Omega))$ with $||\theta||_2 = ||l||$ such that $l(f) = (f, \theta)$ for all $f \in \mathcal{F}(L^2(\Omega))$.

Now $\{\psi_j\}$ is an orthonormal basis of $L^2(\Omega)$ so $\{\hat{\psi}_j\}$ is an orthonormal basis of $\mathcal{F}(L^2(\Omega))$. Therefore by Parseval's identity

$$||l||^2 = ||\theta||^2 = \sum_{j=1}^{\infty} |(\hat{\psi}_j, \theta)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}_j(\xi)|^2.$$

To see that our bound on $||l||$ is attained take $f(x) = e^{2\pi i \xi \cdot x}$. Then $|\hat{f}(\xi)| = \left|\int_\Omega f(x) e^{-2\pi i \xi \cdot x} dm\right| = \left|\int_\Omega dm\right| = m(\Omega)$. Also $||\hat{f}||_2 = ||f||_2 = \left(\int_\Omega \left|e^{2\pi i \xi \cdot x}\right|^2 dx\right)^{1/2} = \left(\int_\Omega dx\right)^{1/2} = m(\Omega)^{1/2}$ so $|l(\hat{f})|/||\hat{f}||_2 = m(\Omega)^{1/2}$ and therefore $||l|| = m(\Omega)^{1/2}$ and our result follows.

(6) Let c be the set of all sequences $\{x_j\}_{j=1}^{\infty}$ with $x_j \in \mathbb{C}$, for which the limit $\lim_{j\to\infty} x_j$ exists.
 (a) Prove that c is a closed subspace of l^{∞}.
 (b) It follows from a) that c, with the l^{∞} norm, is a Banach space. Find an explicit way of characterizing all continuous linear functionals on c.
 (c) Let $l \in c^*$. Is it always possible to find $x \in c$ such that $||x|| = 1$ and $||l|| = |l(x)|$?

 (a) If $x \in c$ then the sequence has a limit so is bounded so is in l^{∞}. If $x^{(1)}$ and $x^{(2)}$ are in c then as $j \to \infty$, then there exist $d^{(1)}, d^{(2)} \in \mathbb{C}$ such that $x_j^{(i)} \to d^{(i)}$ for $i = 1, 2$. So $x^{(1)} + x^{(2)}$ is a sequence that has a limit $d^{(1)} + d^{(2)}$ so is in c. Similarly, for $k \in \mathbb{C}$, $kx^{(1)}$ is a sequence with limit $kd^{(1)}$ so is in c. Therefore c is a linear subspace.
 Next say $x^{(n)} \in c$ and $x^{(n)} \to y$ in l^{∞} as $n \to \infty$. We must show that $y \in c$ so that c is closed. Now $\left|\left|x^{(n)} - y\right|\right|_{\infty} \to 0$ as $n \to \infty$ means that $\sup_{j\in\mathbb{N}} |x_j^{(n)} - y_j| \to 0$ as $n \to \infty$ so for all $\epsilon > 0$ we can find n large enough that $|x_j^{(n)} - y_j| < \epsilon/3$ for all j.
 Next we show y is Cauchy: for all $\epsilon > 0$

$$\begin{aligned} |y_j - y_k| &\leq |y_j - x_j^{(n)}| + |x_j^{(n)} - x_k^{(n)}| + |x_k^{(n)} - y_k| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

 for sufficently large j, k. Here in the first line we used the triangle inequality and an n as mentioned in the paragraph above. Then the bound on the first and third terms of the sum comes from above and the bound on the second term is because $x^{(n)} \in c$ so is Cauchy so for large enough j, k the difference is as small as we wish.
 Therefore $y \in c$ and c is a closed subspace.
 (b) We claim that $c^* = l^1$ when we think of elements $a \in l^1$ as functionals acting by $a(x) = \sum_{k=1}^{\infty} a_k x_k$ on $x \in c$. First we check that such an a is a bounded linear functional. First see that for $x \in c$, $a(x) \in \mathbb{C}$ because

$$|a(x)| \leq \sum_{k=1}^{\infty} |a_k||x_k| \leq ||x||_{\infty} \sum_{k=1}^{\infty} |a_k| = ||x||_{\infty} ||a||_1 < \infty$$

 so the sum converges absolutely. It is linear because sums are linear, for bounded the above calculation shows us that $|a(x)| \leq ||a||_1 ||x||_{\infty}$ so a is bounded as an operator and $||a||_{c\to c} \leq ||a||_1$.
 Now we show an arbitrary $l \in c^*$ must be of this form. If $x \in c$ then $l(x) = l(\sum_{k=1}^{\infty} x_k e_k) = \sum_{k=1}^{\infty} x_k l(e_k)$. So we define a by $a_k = l(e_k)$ and aim to show it is in l^1. Now define $b^{(n)} = \sum_{k=1}^{n} \operatorname{sgn}(a_k) e_k$ which converges to zero as $k \to \infty$ so is in c. Then $l(b^{(n)}) = \sum_{k=1}^{n} a_k \operatorname{sgn}(a_k) = \sum_{k=1}^{n} |a_k| \to ||a||_1$ as $n \to \infty$. But $|l(b^{(n)}| \leq ||l|| \left|\left|b^{(n)}\right|\right|_{\infty} = ||l||$ for all n so $||a||_1 \leq ||l|| < \infty$ because l is bounded. Then $l(x) = \sum_{k=1}^{\infty} x_k l(e_k) = \sum_{k=1}^{\infty} a_k x_k$ for $a \in l^1$.
 (c) No. First note that from our argument above if $l \in c^*$ is the action of $a \in l^1$ then $||l|| \leq ||a||_1$ and also $||a||_1 \leq ||l||$ so the norms must be equal.
 Now take $a = (1, -1/2, 1/4, -1/8, \ldots)$. Then $||a||_1 = 1 + 1/2 + 1/4 + \ldots = 2 < \infty$ so $a \in l^1$ and $||l|| = 2$. Now if $||x||_{\infty} = 1$ and $|l(x)| = 2$ then $|x_k| \leq 1$ for all k and $|\sum_{k=1}^{\infty} a_k x_k| = 2$. But

$$\left|\sum_{k=1}^{\infty} a_k x_k\right| \leq \sum_{k=1}^{\infty} |a_k||x_k| \leq \sum_{k=1}^{\infty} |a_k| = 2$$

 with equality only if $\operatorname{sgn}(a_k x_k)$ is constant and $|x_k| = 1$ for all k.
 So if $|l(x)| = ||l||$ then $x = (1, -1, 1, -1, \ldots)$ or $x = (-1, 1, -1, 1, \ldots)$. But neither of these are in c (they don't converge) so this is not possible.