

# Bordism

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## 1 Introduction

Two closed manifolds are bordant if there exists a compact manifold whose boundary is the disjoint union of those two manifolds. This definition may seem somewhat arbitrary when stated this way but actually leads to a theory that is both very powerful (complex cobordism is universal for cohomology theory on smooth manifolds) and has surprising applications (as Atiyah says in [1], “a topological Quantum Field Theory is a functor from the category of  $n$ -manifolds to the category of complex vector spaces, where the morphisms of manifolds are *cobordisms*”).

We give a basic introduction to unoriented bordism. This is a homology theory that is no more powerful than singular homology theory with  $\mathbb{Z}/2\mathbb{Z}$  coefficients but it lets us see the basic constructions of (co)bordism theory clearly. The one major difference that we will see is that bordism is an *extraordinary* homology theory, that is the homology of a point is not zero.

## 2 The geometric definitions

**Definition 2.1.** A *closed manifold* is a compact manifold without boundary. A *compact manifold* is a compact manifold with boundary.

**Lemma 2.2** (Collaring Lemma). *For a smooth manifold with boundary,  $B^{n+1}$ , there exists an open set  $U \subseteq B^{n+1}$  containing  $M^n = \partial B^{n+1}$  and a diffeomorphism  $\phi: U \rightarrow M^n \times [0, 1)$  with  $\phi(x) = (x, 0)$  for all  $x$  in  $M^n$ .*

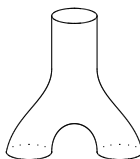
*Proof.* Omitted. Reference [2] recommends seeing [3]. □

**Definition 2.3.** The *connected union* of  $M_1^n$  and  $M_2^n$  is

$$M_1^n \# M_2^n = M_1^n \cup_{S^{n-1}} M_2^n.$$

**Definition 2.4.** Two closed manifolds  $M_1^n$  and  $M_2^n$  are *bordant* if there exists a compact manifold  $B^{n+1}$  with  $\partial B = M_1 \cup M_2$ . This is denoted  $M_1 \sim M_2$ .

**Example 2.5.** The ‘pair of trousers’:  $S^1 \cup S^1 \sim S^1$ . Pictures are thanks to xypic and [4].

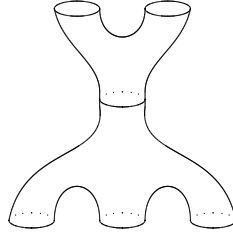


### 3 The bordism ring

In this section we expand upon the discussion of [2] to define the bordism ring.

**Proposition 3.1.** *Bordism is an equivalence relation on the class of closed  $n$ -dimensional manifolds.*

*Proof.* To see  $\sim$  is reflexive note that  $M \times [0, 1]$  is a compact  $n + 1$  dimensional manifold with boundary  $M \cup M$ . Symmetry of the relation is clear from the definition. Transitivity seems reasonable from the following kind of diagram:



To prove it we assume that  $M_1^n \sim M_2^n$  and  $M_2^n \sim M_3^n$ . Then there exists  $B_1^{n+1}$  and  $B_2^{n+1}$  compact manifolds with  $\partial B_1 = M_1 \cup M_2$  and  $\partial B_2 = M_2 \cup M_3$ . Let  $B = B_1 \cup_{M_2} B_2$ , this is a compact  $n + 1$ -dimensional manifold and  $\partial B = M_1 \cup M_3$ .  $\square$

Now denote the equivalence class of  $M^n$  by  $[M]$  and let  $\Omega_n$  be the collection of all such classes of closed  $n$ -dimensional manifolds.

**Proposition 3.2.** *The collection  $\Omega_n$  has an abelian group structure with the group operation*

$$[M_1] + [M_2] = [M_1 \cup M_2].$$

*Proof.* First we show that the operation is well defined: If  $M_1 \sim M'_1$  and  $M_2 \sim M'_2$  are closed  $n$ -dimensional manifolds then there exists compact  $n + 1$ -dimensional manifolds  $B_1$  and  $B_2$  with  $\partial B_1 = M_1 \cup M'_1$  and  $\partial B_2 = M_2 \cup M'_2$ . Then  $B_1 \cup B_2$  is a compact  $n + 1$ -dimensional manifold with  $\partial(B_1 \cup B_2) = (M_1 \cup M'_1) \cup (M_2 \cup M'_2) = (M_1 \cup M_2) \cup (M'_1 \cup M'_2)$ . So  $M_1 \cup M_2 \sim M'_1 \cup M'_2$ .

Now the operation is commutative because  $M_1 \cup M_2 = M_2 \cup M_1$  so the two are certainly bordant. Similarly it is associative because taking unions is associative. The identity for the operation is the equivalence class of the empty manifold, that is  $[\emptyset] = 0$ , because

$$[M] + [\emptyset] = [M \cup \emptyset] = [M].$$

We show that  $2[M] = 0$  for all closed manifolds so that then  $-[M] = [M]$  exists. But this is true because  $M \times [0, 1]$  can be thought of as having boundary  $(M \cup M) \cup \emptyset$ .  $\square$

Now we can define

$$\Omega_O = \sum_{n \geq 0} \Omega_n.$$

**Proposition 3.3.** *The bordism ring,  $\Omega_O$ , is a graded unital commutative ring with multiplication of homogenous elements given by*

$$[M_1^m] \cdot [M_2^n] = [M_1^m \times M_2^n].$$

*Proof.* We first show that this operation is well defined in the first coordinate: if  $M_1 \sim M'_1$  then there exists  $B_1$  a compact manifold with  $\partial B_1 = M_1 \cup M'_1$ . Then  $\partial(B_1 \times M_2) = (\partial B_1 \times M_2) \cup (B_1 \times \partial M_2) = ((M_1 \cup M'_1) \times M_2) \cup (B_1 \times \emptyset) = (M_1 \times M_2) \cup (M'_1 \times M_2)$ . So  $[M_1 \times M_2] = [M'_1 \times M_2]$ . A similar argument shows that the operation is well defined in the second coordinate and then if also  $M_2 \sim M'_2$  we see that  $[M_1 \times M_2] = [M'_1 \times M_2] = [M'_1 \times M'_2]$  so the multiplication of homogenous elements is well defined.

Next note that  $M_1 \times M_2 = M_2 \times M_1$  so the multiplication is commutative. Similarly associativity follows from associativity of manifold multiplication. The multiplicative identity is the equivalence class of a manifold made up of one point. We can see this because

$$[M] \cdot [\{0\}] = [M \times \{0\}] = [M].$$

To see that multiplication distributes over addition we calculate

$$\begin{aligned} [M_1] ([M_2] + [M_3]) &= [M_1 \times (M_2 \cup M_3)] = [(M_1 \times M_2) \cup (M_1 \times M_3)] \\ &= [M_1 \times M_2] + [M_1 \times M_3] = [M_1] \cdot [M_2] + [M_1] \cdot [M_3]. \end{aligned}$$

Next we check that the multiplication is graded. The product  $M_1^m \times M_1^n$  is an  $(m+n)$ -dimensional manifold so we see  $[M_1^m] \cdot [M_1^n] \in \Omega_{m+n}$ .

So we can now extend the multiplication to the entire direct sum by writing elements as sums of homogenous elements and distributing. This is well defined because the multiplication of homogenous elements is distributive.  $\square$

Now let us consider which manifolds are bordant to each other. A dimension zero closed manifold is just a discrete set, so because it is compact it is a finite set of points. Now the set of  $2k$  points is bordant to the empty set as can be seen by considering the compact 1-manifold formed by the disjoint union of  $k$  line segments. Similarly the set of  $2k+1$  points is bordant to 1 point by the manifold of  $k+1$  lines. So  $\Omega_0 \cong \mathbb{Z}/2\mathbb{Z}$ .

A dimension one closed manifold is diffeomorphic to a circle which is bordant to the empty set (as pictured below). So  $\Omega_1 \cong 0$ .



By the classification of surfaces a dimension two closed manifold is diffeomorphic to the connected sum of  $g \geq 0$  toruses or of  $k \geq 1$  real projective planes. The solid torus with  $g$  holes gives a bordism between the torus with  $g$  holes and the empty set. Non-orientable surfaces seem more complicated: the result we cite below says that they are all bordant to  $\mathbb{RP}^2$ .

Beyond this point similar analyses become complicated/impossible. The final answer is known and is proved by converting the question to one in homotopy theory and using spectral sequences.

**Theorem 3.4** (Thom, 1954). *The bordism ring is given by*

$$\Omega_O \cong (\mathbb{Z}/2\mathbb{Z})[a_2, a_4, a_5, \dots, a_n, \dots; n \neq 2^j - 1].$$

Moreover  $a_{2k} = [\mathbb{RP}_{2k}]$ .

*Proof.* Omitted. See [6] for the original proof or [5] for a shorter proof, in English.  $\square$

## 4 Bordism as a homology theory

Fix a topological space  $X$  with a subspace  $A$ . We aim to develop a homology theory based on maps of manifolds into  $X$  much as singular homology is based on maps of simplices into  $X$ . Again our main reference is [2].

**Definition 4.1.** A *singular manifold* in  $(X, A)$  is a pair  $(M^n, f)$  where  $M$  is a compact  $n$ -dimensional manifold and  $f: (M, \partial M) \rightarrow (X, A)$  is a continuous map.

**Definition 4.2.** A singular manifold is said to *bord* if there exists a compact manifold  $B^{n+1}$  such that  $M \subseteq \partial B$  and a map  $F: B \rightarrow X$  such that  $F|_M = f$  and  $F(x) \in A$  for any  $x \in \partial B \setminus M$ .

Given two singular manifolds  $(M_1^n, f_1)$  and  $(M_2^n, f_2)$  we can define their *disjoint union* to be  $(M_1 \sqcup M_2, f_1 \sqcup f_2)$ .

**Definition 4.3.** The two singular manifolds  $(M_1^n, f_1)$  and  $(M_2^n, f_2)$  are *bordant* if their disjoint union bords.

The next three propositions have proofs similar to those we saw for bordism of manifolds and for that reason their proofs are omitted.

**Proposition 4.4.** *Bordism of singular manifolds is an equivalence relation.*

**Proposition 4.5.** *The set of all bordism classes of singular  $n$ -manifolds,  $O_n(X, A)$ , is an abelian group with addition given by disjoint union.*

**Proposition 4.6.** *The direct sum  $O_*(X, A) = \sum_{n \geq 0} O_n(X, A)$  is a graded module over the bordism ring  $\Omega_O$  with multiplication defined by  $[M^n, f][L^m] = [M \times L, g]$  where  $g(x, y) = f(x)$ .*

**Proposition 4.7.** *The transformation  $O_*$  is a functor from the category of pairs of topological spaces to the category of  $\Omega_O$ -modules with a map  $\phi: (X_1, A_1) \rightarrow (X_2, A_2)$  inducing a map  $\phi_*: O_*(X_1, A_1) \rightarrow O_*(X_2, A_2)$  given by  $\phi_*[M, f] = [M, \phi f]$ .*

*Proof.* If  $i: (X_1, A_1) \rightarrow (X_1, A_1)$  is the identity map then  $i_*[M, f] = [M, if] = [M, f]$  so  $i_*$  is the identity on  $O_*(X_1, A_1)$ . If  $\phi: (X_1, A_1) \rightarrow (X_2, A_2)$  and  $\psi: (X_2, A_2) \rightarrow (X_3, A_3)$  then  $(\phi\psi)_*[M, f] = [M, \phi\psi f] = \phi_*[M, \psi f] = \phi_*\psi_*[M, f]$  so  $(\phi\psi)_* = \phi_*\psi_*$  as required.  $\square$

Now define a homomorphism  $\partial: O_n(X, A) \rightarrow O_{n-1}(A, \emptyset)$  by  $\partial[M^n, f] = [\partial M, f|_{\partial M}]$ . This is a natural transformation if  $\partial\phi_* = \phi_*\partial$ . But  $\partial\phi_*[M, f] = \partial[M, \phi f] = [\partial M, (\phi f)|_{\partial M}] = [\partial M, \phi f|_{\partial M}] = \phi_*[\partial M, f|_{\partial M}] = \phi_*\partial[M, f]$  as required.

We aim to show (although we will omit some proofs) that  $O_*$  along with  $\partial$  satisfies the Eilenberg–Steenrod axioms for a homology theory.

**Proposition 4.8** (Homotopy axiom). *If  $f \simeq g: (X_1, A_1) \rightarrow (X_2, A_2)$  then  $f_* = g_*: O_*(X_1, A_1) \rightarrow O_*(X_2, A_2)$ .*

*Proof.* Omitted. See [2].  $\square$

**Lemma 4.9.** *Let  $V^n$  be a regular submanifold with boundary of a closed manifold  $M^n$ . Then if  $f: M \rightarrow X$  is a map with  $f(M \setminus V^\circ) \subseteq A$  then  $[M, f] = [V, f|_V]$ .*

*Proof.* Omitted. See [2]. □

**Proposition 4.10** (Exactness axiom). *For every pair  $(X, A)$  we have an exact sequence*

$$\cdots \xrightarrow{\partial} O_n(A) \xrightarrow{i_*} O_n(X) \xrightarrow{j_*} O_n(X, A) \xrightarrow{\partial} O_{n-1}(A) \xrightarrow{i_*} \cdots$$

where  $i: A \rightarrow X$  and  $j: (X, \emptyset) \rightarrow (X, A)$  are inclusions.

*Proof.* First we show  $\ker \partial = \text{im } J_*$ . Notice that for  $[M, f] \in O_n(A)$  we must have  $\partial M = \emptyset$  and therefore  $\partial j_*[M, f] = \partial[M, jf] = [\partial M, jf|_{\partial M}] = [\emptyset, \emptyset] = 0$  so  $\text{im } j_* \subseteq \ker \partial$ . For the other direction say  $[C, f] \in \ker \partial \subseteq O_n(X, A)$ . Then  $(\partial C, f|_{\partial C})$  bords in  $(A, \emptyset)$  so there exists  $L^n$  and a map  $g: L \rightarrow A$  with  $\partial L = \partial C$  and  $g|_{\partial C} = f$ . Now let  $M = C \cup_{\partial C} L$  be the closed manifold created by gluing  $L$  and  $C$  along their common boundary and define  $F: M \rightarrow X$  by  $F(x) = f(x)$  if  $x \in C$  and  $F(y) = g(y)$  if  $y \in L$ . Then  $[M, F] \in O_n(X)$  and  $j_*[M, F] = [M, jf] = [C, jf|_C] = [C, f]$  where the second equality is by Lemma 4.9.

To see  $\text{im } i_* = \ker j_*$  we first notice that for  $[M, f] \in O_n(A)$  we have  $f: M \rightarrow A$ . Then  $j_* i_*[M, f] = [M, jif]$  and  $jif(M) \subseteq A$  so by Lemma 4.9 with  $V = \emptyset$  we have  $[M, jif] = [\emptyset, jif|_{\emptyset}] = 0$  and  $\text{im } i_* \subseteq \ker j_*$ . I haven't figured out the other direction.

To see  $\text{im } \partial = \ker i_*$  see that for  $[M, f] \in O_n(X, A)$  we have  $i_* \partial[M, f] = [\partial M, if|_{\partial M}]$ . But this bords in  $(X, \emptyset)$  since we can take  $B = M$  and  $F = f$  and then  $\partial B = \partial M$  and  $F|_{\partial M} = f$  so  $i_* \partial[M, f] = 0$  and  $\text{im } \partial \subseteq \ker i_*$ . I haven't figured out the other direction. □

**Lemma 4.11.** *Suppose  $P$  and  $Q$  are closed disjoint subsets of the compact smooth manifold  $M^n$ . Then there exists a smooth submanifold  $M_1^n \subset M$  with  $P \subseteq M_1$  such that  $M_1 \cap Q = \emptyset$  and  $M_1$  is closed in  $M$ .*

*Proof.* Omitted. See [2]. □

**Proposition 4.12** (Excision axiom). *If  $U$  is an open subset of  $X$  whose closure is contained in the interior of  $A$  then the inclusion  $i: (X \setminus U, A \setminus U) \rightarrow (X, A)$  induces an isomorphism  $i_*: O_n(X \setminus U, A \setminus U) \cong O_n(X, A)$ .*

*Proof.* Omitted. See [2]. □

**Proposition 4.13** (Additivity axiom). *If  $X = \sqcup X_\alpha$  then  $O_n(X) \cong \oplus O_n(X_\alpha)$ .*

*Proof.* If we take an element in the direct sum then it is a finite collection of singular manifolds mapping into different  $X_\alpha$ . Take the disjoint union of each of the manifolds and their maps to form an element of  $O_n(X)$ . In the other direction we can take  $(M, f) \in O_n(X)$  and split it up into a finite number of  $M_\alpha$  that map onto  $X_\alpha$  which gives a map into the direct sum. □

**Proposition 4.14** (Failure of the dimension axiom). *The  $n$ -th bordism group of a point is given by  $O_n(p) \cong \Omega_{O,n}$ . In particular it is not always zero for  $n > 0$ .*

*Proof.* The  $n$ -th bordism group is  $O_n(p) = O_n(p, \emptyset) = \{[M^n, f] \mid f: (M, \partial M) \rightarrow (p, \emptyset)\}$ . But this means that  $f$  must take every point to  $p$  and that  $\partial M = \emptyset$  so  $M$  is closed. Beyond this there are no restrictions on  $M$  – every closed manifold maps to a point. Now writing  $f$  for the map that takes all points to  $p$  we have  $[M_1, f] = [M_2, f]$  iff  $[M_1 \sqcup M_2, f]$  bords iff there exists  $B$  with  $M_1 \sqcup M_2 \subseteq \partial B$  such that  $F: B \rightarrow p$  with  $F: M \setminus \partial B \rightarrow \emptyset$  iff  $M_1 \sqcup M_2 = \partial B$  iff  $[M_1] = [M_2]$  in  $\Omega_n$ . So we see that  $O_n(p) = \Omega_n$  as sets. Also we see that  $[M_1, f_1] + [M_2, f_2] = [M_1 \sqcup M_2, f]$  so addition is the same in both groups and  $O_n(p) \cong \Omega_n$ .

To see that the dimension axiom is not satisfied notice that  $O_2(p) \cong \Omega_{O,2} \cong \mathbb{Z}/2\mathbb{Z} \neq 0$ .  $\square$

## 5 Conclusion

We have merely given the definition of a homology theory based on the idea of bordism, which is an extraordinary homology theory. Our main source [2] goes on to look at complex cobordism and show that it is universal.

## References

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