Hypermap-Homology Quantum Codes

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May 10, 2013
Quantum computing

Figure 1: A 128-qubit superconducting adiabatic quantum optimization processor from D-Wave Systems Inc.
Quantum states and Pauli matrices

- We work with a system of $n$ qubits, the state of which is given by an element $|\psi\rangle \in \mathcal{H}_n = (\mathbb{C}^2)^\otimes n$.

- We insist that $|\psi\rangle = \sum_{\vec{i} \in \mathbb{F}_2^n} a_{\vec{i}} |\vec{i}\rangle$ is normalized so that $\sum_{\vec{i} \in \mathbb{F}_2^n} |a_{\vec{i}}|^2 = 1$ and consider states differing by $e^{i\theta}$ with $\theta \in \mathbb{R}$ to be equivalent.

- Define the Pauli operators $X, Y, Z$ by

  $X(a|0\rangle + b|1\rangle) = a|1\rangle + b|0\rangle,$

  $Z(a|0\rangle + b|1\rangle) = a|0\rangle - b|1\rangle$

  and $Y = iXZ$.

- We think of $I, X, Y$ and $Z$ as errors that can occur to a single qubit.
Quantum errors and error correction

- Assume we have encoded a $k$-qubit state into an $n$-qubit state $|\psi\rangle$.
- Define the group

$$G_n = \left\{ c \bigotimes_{i=1}^{n} A_i \mid c \in \{ \pm 1, \pm i \}, A_i \in \{ I, X, Y, Z \} \right\}.$$  

- Assume that an error $E \in G_n$ occurs so that the state is now $E |\psi\rangle$. To correct the error we need to measure $E |\psi\rangle$ in such a way that we only learn $E$ not $|\psi\rangle$.

- A quantum (error-correcting) code is a method for encoding states and correcting errors. A quantum code has parameters $[n, k, d]$ where a code with minimum distance $d$ can correct any errors on at most $\lfloor (d - 1)/2 \rfloor$ qubits.
CSS codes

- A CSS (Calderbank-Shor-Steane) code is defined by two binary matrices $H_X$ and $H_Z$ such that $H_X H_Z^T = 0$.

- We think of $H_X$ and $H_Z$ as being parity check matrices for classical codes $C_X$ and $C_Z$. Thus

$$C_X = \ker H_X, \quad C_Z = \ker H_Z, \quad C_X^\perp = \im H_X^T, \quad C_Z^\perp = \im H_Z^T.$$ 

- It can be shown that a CSS code has parameters $[n, k, d]$ where $H_X$ and $H_Z$ have $n$ columns,

$$k = n - \dim C_X^\perp - \dim C_Z^\perp$$

and

$$d = \min\{\wt(c) : c \in (C_X \setminus C_Z^\perp) \cup (C_Z \setminus C_X^\perp)\}.$$
Some families of CSS codes

- Calderbank and Shor [CS96] showed that there exist a family of CSS codes with \( d \sim c_1 n \) and \( k \sim c_2 n \).
- Codes using sparse matrices are of interest because they can be efficiently decoded. However there are no known sparse CSS codes with performance as good as Calderbank and Shor’s codes.
- Most practically well-performing families of sparse codes (for example MacKay, Mitchison and McFadden’s ‘bicycle codes’ [MMM04]) have minimum distances that are bounded above as \( n \) increases.
- Codes based on homological ideas such as embedding graphs on surfaces have achieved \( d \sim c \sqrt{n} \) or \( d \sim c \sqrt{n \log n} \). The first code of this type was Kitaev’s toric code [Kit97].
Kitaev’s toric code

- Embed an $m \times m$ grid on a torus.

Figure 2: An example with $m = 4$. 
An $m \times m$ grid on a torus
Constructing the toric code

- Consider cellular $\mathbb{F}_2$-homology of this embedded graph. This gives us a chain complex

\[ \mathcal{F} \xrightarrow{\partial_2} \mathcal{E} \xrightarrow{\partial_1} \mathcal{V}. \]

- Take $H_X = [\partial_1]$ and $H_Z = [\partial_2]^T$. Then $H_X H_Z^T = 0$.

- The number of edges is $n = 2m^2$.

- The first homology group is

\[ H_1 = \ker \partial_1 / \text{im} \partial_2 \cong C_X / C_Z^\perp. \]

- Thus $k = n - \dim C_X^\perp - \dim C_Z^\perp = \dim(C_X / C_Z^\perp) = 2$. 
The dual of an $m \times m$ grid on a torus

- The first cohomology group is $H^1 \cong C_{\mathbb{Z}} / C_{\mathbb{X}}^\perp$, which is also the homology of the Poincaré dual graph, pictured below for $m = 4$. 
The minimum distance $d$ is the minimum length of a non-boundary cycle in the graph and its dual graph. So $d = m$.

So we have an $[n, k, d]$ code with $n = 2m^2$, $k = 2$ and $d = m$.

In particular, $d = \sqrt{n/2}$. 
Hypergraphs

- A hypergraph is a generalization of a graph where an edge can be connected to more than 2 vertices.
- A graph:
Hypergraphs

- A hypergraph is a generalization of a graph where an edge can be connected to more than 2 vertices.
- The hypergraph version of the graph:
Hypergraphs

- A hypergraph is a generalization of a graph where an edge can be connected to more than 2 vertices.
- A hypergraph that is not a graph:
Hypergraphs

- A hypergraph is a generalization of a graph where an edge can be connected to more than 2 vertices.
- A hypergraph that is not a graph:

![Hypergraph Diagram]

- Combinatorially we think of a hypergraph as a pair of partitions $V$ and $E$ of $\{1, \ldots, n\}$ where the numbers 1 to $n$ label half-edges we call darts.
Hypermaps

- A hypermap is an embedding of the bipartite graph representation of a hypergraph into a surface. We write the dart’s label counterclockwise of the dart with respect to rotation about edges.

- The picture below is a hypergraph embedded on a torus.
Combinatorial Hypermaps

- A combinatorial hypermap is a pair of permutations $\sigma, \alpha \in S_n$ such that $\langle \sigma, \alpha \rangle$ is transitive. For a survey of hypermaps from this point of view see [CM92].
- From a topological hypermap define $\sigma$ to rotate darts counterclockwise around vertices and $\alpha$ to rotate darts clockwise around edges. Then $\alpha^{-1}\sigma$ rotates darts clockwise around faces.

The next slide shows the combinatorial hypermap corresponding to the example we saw earlier.
We have

$$\sigma = (1 \ 8 \ 3 \ 6)(2 \ 5 \ 4 \ 7), \quad \alpha = (1 \ 2 \ 3 \ 4)(5 \ 6 \ 7 \ 8)$$

and from this we can calculate

$$\alpha^{-1}\sigma = (1 \ 7)(2 \ 8)(3 \ 5)(4 \ 6).$$
The dual hypermap

- Combinatorially, the dual of the hypermap \((\sigma, \alpha)\) is 
  \((\sigma', \alpha') = (\alpha^{-1}\sigma, \alpha^{-1})\).
- Topologically, to create the dual hypermap \(H^*\) of \(H\):
  1. put a vertex of \(H^*\) inside each face of \(H\),
  2. draw a dart of \(H^*\) going from each vertex of \(H^*\) to edges around the corresponding face of \(H\),
  3. move labels from \(H\) to \(H^*\) without crossing darts in either hypermap,
  4. give the surface the opposite orientation.
- The next slide shows an example of a hypermap in black with its dual hypermap in red.
Hypermap homology

- Denote the darts labeled 1, \ldots, n by \(w_1, \ldots, w_n\) and define an \(\mathbb{F}_2\)-chain complex

\[
\mathcal{F} \xrightarrow{d_2} \mathcal{W} \xrightarrow{d_1} \mathcal{V}
\]

by \(d_2(f) = \sum_{i \in f} w_i\) and \(d_1(w_i) = v_{\exists i} + v_{\exists \alpha^{-1}(i)}\).

- Also define \(\iota: \mathcal{E} \to \mathcal{W}\) by \(\iota(e) = \sum_{i \in e} w_i\) extended linearly.

- Notice that \(d_2\) and \(\iota\) are injective, \(d_1 \circ d_2 = 0\) and \(d_1 \circ \iota = 0\).

- So we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{d_2} & \mathcal{W} & \xrightarrow{d_1} & \mathcal{V} \\
& \xrightarrow{\partial_2} & \downarrow{p} & \xrightarrow{\partial_1} & \\
& & \mathcal{W}/\iota(\mathcal{E}) & & \\
\end{array}
\]

and it is the \(\partial_i\) boundary operators that we use to define hypermap-homology.
Hypermap homology group

- Define the first homology of the hypermap to be $H_1 = \ker \partial_1 / \text{im} \partial_2$.
- By counting dimensions we can show that $\dim H_1 = 2g$ where $g$ is the genus of the hypermap.
- Considering the ‘non-hypermap’ homology of the embedded bipartite graph representation of the hypermap we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\partial_2} & \mathcal{W}/\iota(\mathcal{E}) & \xrightarrow{\partial_1} & \mathcal{V} \\
\downarrow id & & \downarrow \mu & & \downarrow i_{\mathcal{V}} \\
\mathcal{F} & \xrightarrow{\bar{d}_2} & \mathcal{W} & \xrightarrow{\bar{d}_1} & \mathcal{V} \oplus \mathcal{E}
\end{array}
\]

where $\mu(w_i + \iota(\mathcal{E})) = w_i + w_{\alpha^{-1}(i)}$.
- We can show $\mu_* : \ker \partial_1 / \text{im} \partial_2 \to \ker \bar{d}_1 / \text{im} \bar{d}_2$ is an isomorphism.
Hypermap homology codes

- To create a CSS code from hypermap-homology we need to choose a basis for $\mathcal{W}/i(\mathcal{E})$.
- From now on we choose a basis given by all darts except one for each edge. (I’ll call this a special basis). We will draw darts in the basis with solid lines and darts not in the basis with dotted lines.
- Having chosen a basis for $\mathcal{W}/i(\mathcal{E})$ and using the distinguished bases for $\mathcal{F}$ and $\mathcal{V}$ we form matrices $H_X = [\partial_1]$ and $H_Z = [\partial_2]^T$.
- Since $H_X H_Z^T = 0$ this gives us a CSS code.
- Then $n = |W| - |E|$, $k = 2g$ and

$$d = \min\{\text{wt}(c) : c \in (C_X \setminus C_Z^\perp) \cup (C_Z \setminus C_X^\perp)\}.$$
An example

- The following hypermap based on a graph from [GKN03] gives rise to a [20, 2, 3] code. (The minimum distance was found by computer).
Finding minimum distance

- We can show $\mu : \ker \partial_1 \setminus \text{im } \partial_2 \to \ker \overline{\partial}_1 \setminus \text{im } \overline{\partial}_2$ is a bijection and thus

  $$\text{minwt}(C_X \setminus C_Z^\perp) = \text{minwt}\{\mu^{-1}(x) : x \in \ker \overline{d}_1 \setminus \text{im } \overline{d}_2\}.$$  

- For $C_Z \setminus C_X^\perp$, we consider the dual hypermap. We can show that $\text{minwt}(C_Z \setminus C_X^\perp)$ is given by the minimum weight of non-boundary cycles in weighted classical homology on the dual hypermap where darts in the chosen basis for $\mathcal{W}/\nu(\mathcal{E})$ have weight 1 and darts not in the basis have weight 0.
Square grid hypermap

- Take an $m \times m$ grid hypermap ($m = 4$ in the picture).
This gives us an \([n, k, d]\) code with 
\[n = 2m^2 - m^2/2 = (3/2)m^2\] and \(k = 2\).

A horizontal or vertical classical cycle \(x\) has 
\[\text{wt}(\mu^{-1}(x)) = m.\]

To see that this is the minimum: if \(x \in \ker \bar{d}_1 \setminus \text{im} \bar{d}_2\) then WLOG \(x\) has at least \(m\) horizontal darts. But \(\mu\) takes a dart to one horizontal and one vertical dart so 
\[\text{wt}(\mu^{-1}(x)) \geq m.\]

Thus \(\text{minwt}(C_X \setminus C_Z^\perp) = m.\)
Square grid hypermap parameters

- We can see from the dual that $\text{minwt}(C_Z \setminus C_X^\perp) = m$. 
Square grid hypermap parameters

Thus a square grid hypermap code has parameters $[(3/2)m^2, 2, m]$.

Comparing this to the toric code which has parameters $[2m^2, 2, m]$ we see that we can store the same amount of quantum information with the same error correcting capability using less physical qubits.
Quantum computers need error correction.

CSS codes can be constructed from $\mathbb{F}_2$-homology.

We constructed hypermap-homology codes by generalizing the toric code; instead of embedding a graph we embedded a hypergraph.

We proved results, especially in the case of a special basis, allowing us to compute the parameters $[n, k, d]$.

For an $m \times m$ square grid the hypermap-homology code has better performance than the toric code.
Some open questions

- Can we find families of hypermaps which have better parameters than the square grid hypermap?
- Must hypermap-homology codes with the special basis we have described satisfy $kd^2 < cn$ for some constant $c$?
- Can we analyze hypermap-homology codes with a non-special basis?
The End

- Thankyou!
- Any questions?
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