Why you can't integrate $\exp(x^2)$

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Integration

- There is an algorithm to find antiderivatives of rational functions (partial fractions). This gives functions that are sums of rational functions, logarithms and inverse trigonometric functions.
- But as soon as we try to integrate more complicated functions we run into trouble. What's the integral of e^{x^2}?

Elementary functions

- The integral of e^{x^2} is $\int e^{x^2}\,dx$. So what do we mean that it can't be integrated?
- We want to give a antiderivative that is an elementary function: a function built from a finite number of exponentials, logarithms, constants, and nth roots through addition, subtraction, multiplication and division.
- Note that we can get trigonometric and inverse trigonometric from exponentials and logs using complex numbers:
  - $\sin(x) = (e^{ix} - e^{-ix})/2i$,
  - $\sin^{-1}(x) = -i\ln(ix + \sqrt{1-x^2})$,
  - and so on.
- So we take our constants to be the complex numbers and our elementary functions to be complex valued functions of a real variable, x.

How do we show things can't be done?

- This question of whether a particular restrictive way of giving an answer is possible reminds us of a few other famous problems:
  - constructibility of various geometric objects with ruler and compass,
  - solvability of polynomial equations in tems of radicals.
- Both of these are studied by looking at towers of field extensions. We need to come up with some algebra.

Differential ring

- A *differential ring* is a ring (characteristic zero, commutative, with identity) along with a *derivation*: an additive homomorphism $R \to R$ denoted $r \mapsto r'$ that satisfies $(rs)' = r's + rs'$.
- This gives $1' = (1 \cdot 1)' = 1' \cdot 1 + 1 \cdot 1'$ which implies $1' = 2 \cdot 1'$ and so $1' = 0$.
- A simple induction gives $(r^n)' = nr^{n-1}r'$.
- Our favourite example is $\mathbb{C}[x]$ with the derivation d/dx.

Differential field

- A *differential field* is a field that is a differential ring.
- To find derivation of an inverse start with $(rr^{-1}) = 1$. Taking derivation of both sides we get $r'r^{-1} + r(r^{-1})' = 0$ which gives $(r^{-1})' = -r'/r^2$.
- From this result and the product rule we get the quotient rule

$$\left(\frac{r}{s}\right)' = \frac{r's - rs'}{s^2}.$$

- Our favourite example is $\mathbb{C}(x)$ with the derivation d/dx.

Field of constants

- If K is a differential field then $K_C = \{r \in K \colon r' = 0\}$ is a subfield, the *field of constants*. It contains $\mathbb{Q}$ (with our definition that K is characteristic zero).
- The field extension K/K_C is purely transcendental.
- A differential field extension L/K is said to be *no new constants* if $L_C = K_C$. This is equivalent to not introducing new antiderivatives for elements that already have them.

Integration in finite terms

- Let K be a differential field. An element $\ell \in K$ is a *logarithm* of an element $k \in K$ (or k is an *exponential* of ℓ) if $\ell' = k'/k$.
- A differential field extension L/K is *elementary* if there is a finite sequence of intermediate differential field extensions

$$K = K_0 \subset K_1 \subset \cdots \subset K_n = L$$

  such that $K_{j+1} = K_j(\ell)$ where ℓ is either algebraic over K_j, a logarithm of some element of K_j or an exponential of some element of K_j.
- Any element of L is said to be *elementary*.

Liouville's theorem

Theorem (Liouville, 1835; Rosenlicht, 1961)

Let K be a differential field of characteristic 0 and $\alpha \in K$ be an element with no antiderivative in K. Then α has a antiderivative in some elementary no new constant differential field extension of K if and only if there exists $m \geq 1$, constants $c_1, \ldots, c_m \in K_C$ and elements $\beta_1, \ldots, \beta_m, \gamma \in K$ such that

$$\alpha = \sum_{j=1}^{m} c_j \frac{\beta_j'}{\beta_j} + \gamma'.$$

A useful corollary

Corollary

Suppose $K = E(e^g)$ is a no new constant differential field extension of E for some element $g \in E$. Then fe^g has a antiderivative within some elementary no new constant differential field extension of K if and only if there is an element $a \in E$ such that

$$f = a' + ag'.$$

Proof.

($\Leftarrow$) Multiply by e^g so $fe^g = a'e^g + ag'e^g = (ae^g)'$.
($\Rightarrow$) Omitted. $\square$

Showing e^{x^2} has no antiderivative

- We take $f = 1$ and $g = x^2$ in $E = \mathbb{C}(x)$. Then the corollary says that $fe^g = e^{x^2}$ has an elementary antiderivative if and only if there exists $a \in \mathbb{C}(x)$ such that $1 = a' + 2xa$. Take $a = r/s$ in lowest terms then we have

$$1 = \frac{r's - rs'}{s^2} + 2x\frac{r}{s}.$$

- This gives $s^2 = r's - rs' + 2xrs$ which implies $s(s - 2xr - r') = rs'$.
- So $s \mid s'$ which implies $s' = 0$ and thus s is a constant.
- Now a is a polynomial so $1 = a' + 2xa$ is a contradiction.

What else doesn't have an antiderivative?

- e^{-x^2}
- e^x/x, $1/\log x$
- $\sin(x)/x$
- $1/\sqrt{P(x)}$ for P a monic polynomial of degree ≥ 3 with no repeated roots.
- x^x

Is there a way to tell in general?

- Yes!
- Risch came up with an algorithm in 1970.
- This ‘algorithm’ has some problems:
  - Need to be able to solve polynomial equations.
  - Need to be able to tell whether expressions are equal. It is not known whether there is an algorithm to do this for elementary functions (in fact it is known that for a class of elementary functions that include the absolute value function such an algorithm does not exist).
  - Hard to program. Slow.
- Axiom has implemented some of the negative part of the algorithm. If Axiom says that something can’t be integrated it really can’t.

Examples to try in Axiom

- $$\int \frac{x}{\sqrt{x^4 + 10x^2 - 96x - 71}} \, dx$$
- $$\int \frac{x}{\sqrt{x^4 + 10x^2 - 96x - 72}} \, dx$$
- $$\int \frac{x^2 + 2x + 1 + (3x+1)\sqrt{x + \ln x}}{x\sqrt{x + \ln x}(x + \sqrt{x + \ln x})} \, dx$$

Differential Galois Theory

- We are considering linear differential equations $Y' = AY$.
- There are extensions called Picard-Vessiot extensions that play the role of splitting fields in standard Galois theory.
- The Galois group of an equation $Y' = AY$ is the differential automorphisms of a Picard-Vessiot extension. In general this group is an algebraic group which can be thought of as a matrix Lie group.
- For L a Picard-Vessiot extension of K with Galois group G the Galois correspondence is between *closed* subgroups of G and *differential* subfields $K \subset M \subset L$.

Some results

- We consider differential equations to be solvable if we can find a solution in a tower of field extensions that are algebraic or extensions by exponentials or integrals.
- An equation is solvable in this sense if and only if the connected component of the identity of its Galois group is solvable.
- An equation is solvable by exponentials if and only if its Galois group is diagonalisable.

Is any of this useful?

- Not really...