MATH534B Final Exam · Due Thursday May 15

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(1) Let M be a smooth manifold without boundary, of dimension $m+n+1$. Let ω be an n-form and and η an m-form. Show that

$$\int_M \omega \wedge d\eta = c \int_M d\omega \wedge \eta$$

for some constant c and find c.

By Theorem 12.14(ii) in Lee we have $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^n \omega \wedge d\eta$ so

$$\omega \wedge d\eta = (-1)^n d\omega \wedge \eta - (-1)^n d(\omega \wedge \eta)$$

and by the linearity of the integral

$$\int_M \omega \wedge d\eta = (-1)^n \int_M d\omega \wedge \eta - (-1)^n \int_M d(\omega \wedge \eta).$$

But

$$\int_M d(\omega \wedge \eta) = \int_{\partial M} \omega \wedge \eta = 0$$

where the first equality is by Stokes' theorem and the second is because M has no boundary. Therefore the result follows with $c = (-1)^n$.

(2) Let p be a real homogenous polynomial in n variables. Show that for all $c \neq 0$, the set $\{(x^1, \ldots, x^n) \mid p(x^1, \ldots, x^n) = c\}$ is an embedded submanifold of $\mathbb{R}^n$.

If p is a homogenous polynomial then there exists $m > 0$ (the total power of each term) such that

$$p(tx^1, \ldots, tx^n) = t^m p(x^1, \ldots, x^n) \text{ for all } t \in \mathbb{R}^n.$$

So taking the derivative with respect to t of of both sides we see that

$$\frac{d}{dt}(p(tx^1, \ldots, tx^n) = \sum_{i=1}^n \frac{\partial p}{\partial(tx^i)}(tx^1, \ldots, tx^n)\frac{\partial(tx^i)}{\partial t} = \sum_{i=1}^n x^i \frac{\partial p}{\partial x^i}(tx^1, \ldots, tx^n)$$

is equal to

$$\frac{d}{dt} t^m p(x^1, \ldots, x^n) = m \cdot t^{m-1} p(x^1, \ldots, x^n).$$

So setting $t = 1$ we have shown that

$$\sum_{i=1}^n x^i \frac{\partial p}{\partial x^i}(x^1, \ldots, x^n) = m \cdot p(x^1, \ldots, x^n).$$

Now by the regular level set theorem if we show that $dp_x \neq 0$ for $x \in p^{-1}(c)$ we can conclude that $p^{-1}(c)$ is an embedded submanifold. But $dp_x = 0$ only if $\partial p / \partial x^i = 0$ for all i and then by the formula we have shown $mc = m \cdot p(x) = \sum x^i \cdot 0 = 0$ which contradicts m and c both not being zero. Thus $dp_x \neq 0$ for $x \in p^{-1}(c)$ and we are done.

(3) Let $f\colon \mathbb{D}^2 \to \mathbb{R}^2$ be a continuous function from the disk to the plane. Let γ be the loop $f(\partial\mathbb{D}^2) \subset \mathbb{R}^2$. Let $a \in \mathbb{R}^2$ and suppose that $[\gamma] \in \pi_1(\mathbb{R}^2 \setminus \{a\})$ is nontrivial. Show that there exists an $x \in \mathbb{D}^2$ such that $f(x) = a$.

Assume, seeking a contradiction, that there does not exist $x \in \mathbb{D}^2$ such that $f(x) = a$. Then we have $f\colon \mathbb{D}^2 \to \mathbb{R}^2 \setminus \{a\}$ which induces a homomorphism $f_*\colon \pi_1(\mathbb{D}^2) \to \pi_1(\mathbb{R}^2 \setminus \{a\})$. But $\pi_1(\mathbb{D}^2) = 0$ so this must be the zero homomorphism and $[\gamma] = f_*([\partial\mathbb{D}^2]) = 0$ contradicting our assumption that $[\gamma]$ was nontrivial. Thus there does exist such an x.

(4) (a) Show that $\mathbb{R}^3 \setminus S^1$ deformation retracts onto $S^1 \vee S^2$.
(b) Find an equally nice deformation retract of S^3 with two unlinked circles deleted.

(a) We give a succession of homotopies and explain them below (pictures are on the next page). Let A be the circle removed from $\mathbb{R}^3$ and then

$$\mathbb{R}^3 \setminus A \simeq B^3 \setminus A \simeq S^2 \cup D \simeq (S^2/\sim) \simeq S^2 \vee S^1.$$

In order, these homotopies follow by the operations

(i) Take a ball, B^3, around the center of A that contains A and then deformation retract all points outside the ball to the surface of the ball.

(ii) Let S^2 be the surface of the B^3 described above and D be a diameter of S^2 that passes through the center of A. For a point $x \in B^3$ draw a ray starting from the closest point on A through x and then onto hit whichever comes first from S^2 or D. This defines a deformation retract of $B^3 \setminus A$ onto $S^2 \cup D$: the map is well defined because if there are two points closest to x then $x \in D$ so doesn't get moved. Also D and S^2 are both fixed by the map and it is continuous by construction.

(iii) Collapse the contractible subspace D. Then the new space is S^2 with two points identified.

(iv) The space of S^2 with a string connecting two points is homotopic to $S^2/\sim$ by collapsing the contractible string. Also S^2 with a string connecting two points is homotopic to $S^2 \vee S^1$ by collapsing a contractible arc on the surface of S^2 connecting the two points.

(b) Let A and B be the two unlinked circles on S^3. Then stereographic projection from a point on B takes S^3 minus a point to $\mathbb{R}^3$ and takes A to a circle we shall still call A and B to a line L which doesn't go through A. Then

$$S^3 \setminus (A \cup B) \simeq \mathbb{R}^3 \setminus (L \cup A) \simeq (B^3 \setminus A) \vee S^1 \simeq (S^2 \vee S^1 \vee S^1).$$

Here the second homotopy follows by taking a ball around A and a cylinder around L and then deformation retracting the cylinder to S^1 and at the same time closing off the neck connecting the ball to the cylinder leaving them connected only at one point. The second homotopy is part (a) of the question.

(5) Let M be the Möbius strip, viewed as the square $-1/2 \leq x, y \leq 1/2$. The vertical sides are identified according to up-arrow at $x = -1/2$, down arrow at $x = 1/2$.

(a) Show that the universal covering space $\widetilde{M}$ of M can be identified with the infinite strip $M_\infty = \{(x, y) \mid -\infty < x < \infty, -1/2 \leq y \leq 1/2\}$.

(b) Find the group $G(\widetilde{M})$ of deck transformations, and use it to obtain the fundamental group $\pi_1(M)$.

(c) For what integers n does there exist an n-sheeted covering of the Möbius strip by a Möbius strip?

(a) Notice that M_∞ is contractible so it is simply connected, therefore we only need to show that it is a covering space of M. Define a map $p\colon M_\infty \to M$ by first finding an n such that $x \in n + (-1/2, 1/2]$ and then setting $p(x, y) = (x - n, (-1)^n y)$. This is equivalent to splitting M_∞ into infinitely many Möbius strips oriented in alternate directions as pictured below.

To see this map is continuous note that the map is clearly continuous inside each $(n - 1/2, n + 1/2] \times [-1/2, 1/2]$. For points close to each other but on different sheets the function p maps them to different sides of M which are close together via the identification. For example $p(1/2 - \epsilon, y) = (1/2 - \epsilon, y)$ and $p(1/2 + \epsilon, y) = (-1/2 + \epsilon, -y)$ which are close because $(1/2, y) \sim (-1/2, -y)$ under the identification on M.

To see that p is a covering space map take a covering of M by two open sets, U_1 which includes the identification and U_2 which doesn't. Then clearly $p^{-1}(U_2)$ is a union of disjoint open sets each of which is homeomorphic to U_2 via p and similarly for U_1.

So M_∞ is a simply connected covering space of M, therefore $\widetilde{M} \cong M_\infty$.

(b) Take $f \in G(\widetilde{M})$. Then $f\colon \widetilde{M} \to \widetilde{M}$ and $pf = p$. Now take $x \in n + (-1/2, 1/2]$ and we see that $p(x, y) = (x - n, (-1)^n y)$. If $f(x, y) = (x_1, y_1)$ with $x_1 \in m + (-1/2, 1/2]$ then $pf(x, y) = p(x_1, y_1) = (x_1 - m, (-1)^m y_1)$. So for $pf = p$ we require $x_1 - m = x - n$ and $(-1)^m y_1 = (-1)^n y$, that is $x_1 = x - (n - m)$ and $y_1 = (-1)^{n-m} y$.

So for each $f \in G(\widetilde{M})$ there exists an $m \in \mathbb{Z}$ such that $f(x, y) = (x - m, (-1)^m y)$. Call such a map f_m and we show that all such maps are isomorphisms. Clearly f_m is a homeomorphism (is just a translation horizontally and possible a vertical flip). To see it is an isomorphism we need $pf_m = p$. But for $x \in n + (-1/2, 1/2]$ we see that $pf_m(x, y) = p(x - m, (-1)^m y) = (x - m - (n - m), (-1)^{n-m}(-1)^m y) = (x - n, (-1)^n y) = p(x, y)$. Therefore $G(\widetilde{M}) = \{f_m \mid m \in \mathbb{Z}\}$. Finally we calculate, for $x \in n + (-1/2, 1/2]$ that $f_{m_1} f_{m_2}(x, y) = f_{m_1}(x - m_2, (-1)^{m_2} y) = (x - m_2 - m_1, (-1)^{m_2}(-1)^{m_1} y) = f_{m_1+m_2}(x, y)$ and see that $f_{m_1} f_{m_2} = f_{m_1+m_2}$ so that $G(\widetilde{M}) \cong \mathbb{Z}$.

Therefore $\pi_1(M) = \mathbb{Z}$.

(c) For n odd a covering similar to the one described above gives an n-sheeted covering of M by itself.

For n even the same covering gives a covering of M by the cylinder. If an n sheeted covering of M by M were to exist then this would induce a map p_* such that $p_*(\pi_1(M))$ is an index n subgroup of $\pi_1(M) \cong \mathbb{Z}$. But there is only one index n subgroup of $\mathbb{Z}$, namely $n\mathbb{Z}$. Then the covering by the cylinder C is another n-sheeted cover so its fundamental group pushes forward to an index n subgroup which must be the same $n\mathbb{Z}$. So by Proposition 1.37 in Hatcher we then have that the two covering spaces are isomorphic, in particular that the Möbius strip and the cylinder are homeomorphic. But this is not true so there must not be an n-sheeted covering of M by itself for n even.

Therefore there exists an n-sheeted covering of M by itself if and only if n is odd.

(6) Let C be the cube $-1/2 \le x, y, z \le 1/2$. Let X be the space obtained by rotating the right face by 90° (in the positive x direction) and then identifying it with the left face.

(a) Find the universal covering space $\widetilde{X}$ of X.

(b) Find the group $G(\widetilde{X})$ of deck transformations and use it to obtain the fundamental group of X.

(c) Find a covering of X by a solid torus. How many sheets are there in your covering?

(a) Take an infinitely long bar $\widetilde{X} = \{(x, y, z) \mid -1/2 \le x, y \le 1/2\}$ and define a map $p\colon \widetilde{X} \to X$ that takes a point (x, y, z) with $z \in [n, n+1)$ to the point $(x, y, z-n)$ and then rotates $n\pi/2$ in a counter clockwise direction. Symbolically this is

$$p(x,y,z) = \begin{cases} (x, y, z-n) & \text{for} \quad n \equiv 0 \pmod 4 \\ (x, -y, z-n) & \text{for} \quad n \equiv 1 \pmod 4 \\ (-x, -y, z-n) & \text{for} \quad n \equiv 2 \pmod 4 \\ (-x, y, z-n) & \text{for} \quad n \equiv 3 \pmod 4. \end{cases}$$

and we can also think of splitting $\widetilde{X}$ into copies of X as drawn below.

Then p is clearly a covering space map. Also $\widetilde{X}$ is contractible so simply connected, therefore is the universal cover of X.

(b) If $f \in G(\widetilde{X})$ then it must take a point (x, y, z) to a point with z-coordinate $z + m$ for some $m \in \mathbb{Z}$. For $pf = p$ we need f to rotate $(x, y, z+m)$ by $m\pi/2$ in a clockwise direction. Call this homomorphism f_m then $f_{m_1} f_{m_2}(x, y, z) = (x, y, z + m_1 + m_2)$ rotated $\pi(m_1 + m_1)/2$ in a clockwise direction so $f_{m_1} f_{m_2} = f_{m_1+m_2}$ and $G(\widetilde{X}) \cong \mathbb{Z}$.

(c) Divide the solid torus into four equal quarters and map each quarter to $\widetilde{X}$ by identifying the end circles to each other with a 90° twist and then straightening, squaring up the edges and squashing into a cube.

This is a continuous map and if we cover X by two open sets then the inverse image of each open set is four disjoint open sets homeomorphic to the original so the solid torus is a covering space. This is a four sheeted cover. One way to see this is to take the generator for the the fundamental group of the solid torus (a circle through the center) and map it into X. There it is 4 loops all on top of each other through the center of X. But $\pi_1(X)$ is generated by the loop through the center (since X deformation retracts to the center line with endpoints identified) so $p_*(\pi_1(\text{solid torus})) = 4\mathbb{Z} \le \pi_1(X)$. So this is an index 4 subgroup which means the solid torus is a four sheeted covering space.

(7) Let A be the union of the x and y-axes in $\mathbb{R}^3$ and $X = \mathbb{R}^3 \setminus A$. Define open sets U and V by

$$U = \{(x, y, z) \mid z > 0\} \cup \{(x, y, z) \mid z > -1, (x, y) \notin A\} \text{ and}$$
$$V = \{(x, y, z) \mid z < 0\} \cup \{(x, y, z) \mid z < 1, (x, y) \notin A\}.$$

Use the Mayer–Vietoris sequence for deRham cohomology to find the cohomology groups $H^k(X)$.

Note that $U \cup V = X$. Also $U \cap V$ is the disjoint union of 4 contractible spaces

so

$$H^k(U \cap V) = \begin{cases} \mathbb{R}^4 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k \geq 1. \end{cases}$$

Also U is contractible (imagine squashing everything with $z < 0$ up to $z = 0$ and then pulling everything in to the point $(0, 0, 1)$). Similarly V is contractible so

$$H^k(U) = H^k(V) = \begin{cases} \mathbb{R} & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k \geq 1. \end{cases}$$

Now X is an open subset of a 3-manifold so $H^k(X) = 0$ for $k \geq 4$. Then the Mayer–Vietoris sequence gives us an exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(U \cup V) \longrightarrow H^0(U) \oplus H^0(V) \longrightarrow H^0(U \cap V) \longrightarrow \\ H^1(U \cup V) \longrightarrow H^1(U) \oplus H^1(V) \longrightarrow H^1(U \cap V) \longrightarrow \\ H^2(U \cup V) \longrightarrow H^2(U) \oplus H^2(V) \longrightarrow H^2(U \cap V) \longrightarrow \\ H^3(U \cup V) \longrightarrow H^3(U) \oplus H^3(V) \longrightarrow H^3(U \cap V) \longrightarrow 0. \end{array}$$

Filling in the parts of the diagram we know, we get the exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(X) \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^4 \longrightarrow \\ H^1(X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \\ H^2(X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \\ H^3(X) \longrightarrow 0 \longrightarrow 0 \longrightarrow 0. \end{array}$$

Now if $0 \to A \to 0$ is an exact sequence then $A = \ker 0 = \operatorname{im} 0 = 0$ so we can conclude that $H^2(X) = H^3(X) = 0$. For $H^0(X)$ we notice that X is connected so $H^0(X) = \mathbb{R}$. So we have an exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\phi_1} \mathbb{R}^2 \xrightarrow{\phi_2} \mathbb{R}^4 \xrightarrow{\phi_3} H^1(X) \longrightarrow 0.$$

So we see that ϕ_1 is injective and ϕ_3 is surjective. By the first isomorphism theorem and exactness at $\mathbb{R}^2$ we see that

$$\operatorname{im}\phi_2 \cong \mathbb{R}^2/\ker\phi_2 = \mathbb{R}^2/\operatorname{im}\phi_1 \cong \mathbb{R}^2/\mathbb{R} \cong \mathbb{R}.$$

Then similarly, using the surjectivity of ϕ_3,

$$H^1(X) = \operatorname{im}\phi_3 \cong \mathbb{R}^4/\ker\phi_3 = \mathbb{R}^4/\operatorname{im}\phi_2 \cong \mathbb{R}^4/\mathbb{R} \cong \mathbb{R}^3.$$

Therefore

$$H^k(X) = \begin{cases} \mathbb{R} & \text{for} \quad k=0 \\ \mathbb{R}^3 & \text{for} \quad k=1 \\ 0 & \text{for} \quad k \geq 2. \end{cases}$$

(8) Remove an open disk U from the sphere S^2 and glue a Möbius strip M to the sphere (∂M is glued to ∂U). Call the resulting space X.
  (a) Present X as a Δ-complex.
  (b) Find the simplicial homology groups $H_k(X)$.
  (c) Find the fundamental group of X.
  (d) X is a popular space. Which one?

  (c) With the CW complex for M and X given below with three 0-cells, four 1-cells and two 2-cells

  we see that the 1–skeleton of X looks like

  Thus we calculate that $\pi_1(X_1) = \langle fg, hf^{-1}\rangle$. To see this note that after collapsing a and f the space is $S^1 \vee S^1$ and the loops in the original X_1 corresponding to the two circles are the generators above.
  The relations come from attaching the two 2-cells, starting our loops from $\bullet$. Travelling in a clockwise direction our relations are $aa^{-1}hg$ and $fgfh^{-1}$.
  Therefore $\pi_1(X) = \langle fg, hf^{-1} \mid hg, fgfh^{-1}\rangle$. But the second relation tells us that $hf^{-1} = fg$ so the group is generated by fg. Noting that $h = fgf$ we have $\pi_1(X) = \langle fg \mid fgfg\rangle \cong \mathbb{Z}/2$.
  (d) Recall that we can also think of M as an annulus with the outside edge the boundary and the inside edge identified antipodally. Thus a sphere with the boundary of M glued into a removed disc is homeomorphic to a hemisphere with antipodal points on the boundary identified, i.e. $X \cong \mathbb{RP}^2$.

(a) Now that we know what X is we can write down a Δ-complex for it[1].
The Δ-complex for $x = \mathbb{RP}^2$ is given below as the identifications on a square

(b) To calculate $H_k(X) = H_k(\mathbb{RP}^2)$ we first list the n-chains: $\Delta_0(X) = \mathbb{Z}v + \mathbb{Z}w$, $\Delta_1(X) = \mathbb{Z}a + \mathbb{Z}b$ and $\Delta_2(X) = \mathbb{Z}U + \mathbb{Z}L$.
Then we have the chain complex

$$0 \xrightarrow{\partial_3} \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0$$

with $\partial_3 = \partial_0 = 0$. To calculate the other boundary maps we see

$$\partial_2 \colon U \mapsto a - b - c \text{ and } L \mapsto a - b + c.$$

Also

$$\partial_1 \colon a \mapsto w - v, b \mapsto w - v \text{ and } c \mapsto 0.$$

Now if $mU + nL \in \ker \partial_2$ then $m(a - b - c) + n(a - b + c) = 0$. Thus $m + n = 0$ and $m - n = 0$ so $m = n = 0$ and $\ker \partial_2 = 0$.
Therefore $H_2(X) = \ker \partial_2 / \operatorname{im} \partial_3 = 0/0 = 0$.
If $sa + tb + uc \in \ker \partial_1$ then $s(w - v) + t(w - v) = 0$. Therefore $t = -s$ and u is arbitrary so $\ker \partial_1 = \langle a - b, c\rangle = \langle a - b + c, c\rangle$. The last equality is for later use and follows because we can write $a - b + c = (a - b) + c$ and $a - b = (a - b + c) - c$.
Also $\operatorname{im} \partial_2 = \langle a - b - c, a - b + c\rangle = \langle a - b + c, 2c\rangle$ where the second equality is because $2c = (a - b + c) - (a - b - c)$ and $a - b - c = (a - b + c) - (2c)$.
So $H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \langle a - b - c, c\rangle / \langle a - b - c, 2c\rangle = \langle c \mid 2c\rangle \cong \mathbb{Z}/2$.
Finally $H_0(X) = \ker \partial_0 / \operatorname{im} \partial_1 = \langle v, w\rangle / \langle w - v\rangle = \langle v\rangle \cong \mathbb{Z}$.
Therefore

$$H_k(X) = \begin{cases} \mathbb{Z} & \text{for} \quad k = 0 \\ \mathbb{Z}/2 & \text{for} \quad k = 1 \\ 0 & \text{for} \quad k \geq 2. \end{cases}$$

[1] this probably isn't how you're meant to do the question but I really couldn't figure out how to do it directly.