Section 4.5 Solutions - Max/Min Problems.

1. $x+y=5$ and product $= xy$ so $p(x)$ is product as a function of x $p(x) = x(5-x)$ which is a parabola whose max value occurs at the vertex vertex is $(2.5, 6.25)$. So $(2.5)(2.5) = 6.25$ which is the largest possible product.

2. $x+y=20$ want x^2+y^2 = smallest possible value.
Since $y = 20-x \Rightarrow$ $x^2+(20-x)^2 = x^2+400-40x+x^2$
$= 2x^2-40x+400 = 2(x^2-20x+100-100)+400 \Rightarrow$
$2(x^2-20x+100)-200+400 = 2(x-10)^2+200 = S(x)$
so minimum is vertex $(10, 200)$ so when $x=10=y$
$10^2+10^2 = 200$ is the smallest value

4. a. $y = 2x^2-8x+1$ is a parabola that points up so we look for the lowest point on the graph
$$y = 2(x^2-4x+4-4)+1 = 2(x-2)^2-7$$
So Coordinate is $(2,-7)$

b. $y = -3x^2-4x-9$ is a parabola that points down so we look for the highest point on the graph
$$y = -3\left(x^2+\tfrac{4}{3}x+\tfrac{4}{9}-\tfrac{4}{9}\right)-9 = -3\left(x+\tfrac{2}{3}\right)^2+\tfrac{4}{3}-9$$
$= -3\left(x+\frac{2}{3}\right)^2 - \frac{23}{3}$ so coordinate is $\left(-\frac{2}{3}, -\frac{23}{3}\right)$

c. $h = -16t^2+256t$ points down so looking for max
$$h = -16(t^2-16t+64-64) = -16(t^2-16t+64)+1024$$
$= -16(t-8)^2+1024$ Coordinate $(8, 1024)$

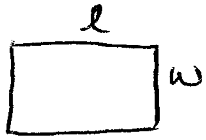
d. $f(x) = 1-(x+1)^2$ points down so max vertex $(-1,1)$

e. $g(t) = t^2+1$ points up so min vertex $(0,1)$

4f. $f(x) = 1000x^2 - x + 100$ Points up so min

$f(x) = 1000\left(x^2 - \frac{1}{1000}x + \frac{1}{4{,}000{,}000} - \frac{1}{4{,}000{,}000}\right) + 100 = 1000\left(x - \frac{1}{2000}\right)^2 - \frac{1}{4000} + 100$

$= 1000\left(1 - \frac{1}{2000}\right)^2 + \frac{399{,}999}{4000}$ Coordinate $\left(\frac{1}{2000}, \frac{399{,}999}{4000}\right)$

6. 

Perimeter $= 80 = 2l + 2w \Rightarrow l = 40 - w$

Area $= lw$ So $A(w) = w(40-w) = -w^2 + 40w$

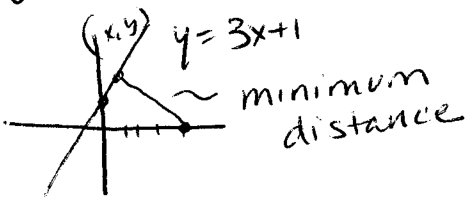
So vertex is max and is $(20, 400)$ So when $l = 20 = w$ cm

the max area is $400\,cm^2$.

10. $h = 512t - 16t^2 = -16(t^2 - 32t + 256 - 256) =$

$-16(t-16)^2 + 3096$ So at 16 seconds the height

of the object is 3096 feet.

14. 

$d = \sqrt{(4-x)^2 + (0-y)^2}$

$= \sqrt{16 - 8x + x^2 + y^2}$

So the value that minimizes d will be the same as that which minimizes $d^2 = 10x^2 - 2x + 17$

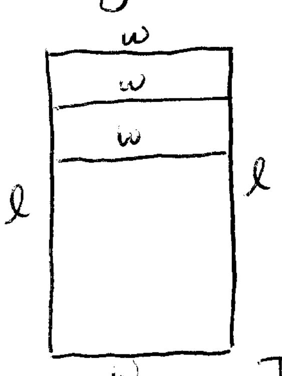
$= \sqrt{16 - 8x + x^2 + (3x+1)^2}$

$= \sqrt{10x^2 - 2x + 17}$

so complete the square $d^2 = 10\left(x^2 - \frac{1}{5}x + \frac{1}{100} - \frac{1}{100}\right) + 17$

$= 10\left(x - \frac{1}{10}\right)^2 + \frac{169}{10}$ Since this is a parabola which opens up with vertex $\left(\frac{1}{10}, \frac{169}{10}\right)$, the minimum occurs at $x = \frac{1}{10}$

and $y = 3\left(\frac{1}{10}\right) + 1 = \frac{13}{10}$ so point is $\left(\frac{1}{10}, \frac{13}{10}\right)$

16. 

There are 1800 feet of fencing so $4w + 2l = 1800$

and so $l = 900 - 2w$

$A = lw = w(900 - 2w) = -2w^2 + 900w$

complete square $A = -2(w^2 - 450w + 50625 - 50625)$

$= -2(w - 225)^2 + 101{,}250$ This is a parabola that points down so max is at $w = 225$ $y = 900 - 2(225) = 450$

So dimensions are 225 meters by 450 meters.

17. Let d be depth of pasture, then $500-2d$ is length parallel to the river so Area $= d(500-2d) = -2d^2+500d$

Completing square $A = -2(d^2 - 250d + 125^2 - 125^2) = -2(d-125)^2 + 31250$

this a parabola pointing down so max is at $d = 125$

and length is $500 - 2(125) = 250$. So dimensions are 125 feet by 250 feet.

24. a) vertex of $y = 2x^2 - 4x + 7$ can be found by completing square

$2(x^2 - 2x + 1 - 1) + 7 \Rightarrow 2(x-1)^2 - 2 + 7 = 2(x-1)^2 + 5$

vertex is $(1,5)$

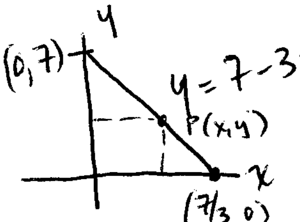
b) $y = 2x^2 - 4x + 7 - x^2 = x^2 - 4x + 7 = (x^2 - 4x + 4 - 4) + 7$

$= (x-2)^2 + 3$ so minimum length of road is $7 - 4 = 3$ miles.

26. $r = kx(a-x) = -kx^2 + kax$. Complete square:

$r = -k\left(x^2 - ax + \frac{a^2}{4} - \frac{a^2}{4}\right) = -k\left(x - \frac{a}{2}\right)^2 + \frac{ka^2}{4}$

which is downward pointing parabola so max at $x = \frac{a}{2}$

31. 

$A = xy = x(7-3x) = -3x^2 + 7x = -3\left(x^2 - \frac{7}{3}x\right)$

$= -3\left(x^2 - \frac{7}{3}x + \frac{49}{36} - \frac{49}{36}\right) = -3\left(x - \frac{7}{6}\right)^2 + \frac{49}{12}$

so the largest possible area is $\frac{49}{12}$ units^2

32. $A = xy = x(mx+b) = mx^2 + bx = m\left(x^2 + \frac{b}{m}x + \frac{b^2}{4m^2} - \frac{b^2}{4m^2}\right)$

$= m\left(x - \frac{b}{2m}\right)^2 - \frac{b^2}{4m}$. Since $m<0$, largest possible area is $-\frac{b^2}{4m}$

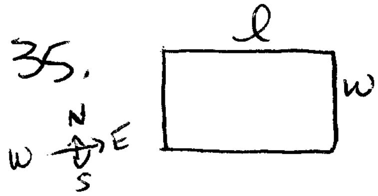
34. area of rectangle is $(x)(2r) \Rightarrow A = 2rx$. Total perimeter $= \frac{1}{4} = 2x + 2\pi r \Rightarrow r = \frac{1-8x}{8\pi}$

$A = 2x\left(\frac{1-8x}{8\pi}\right) = -\frac{2}{\pi}x^2 + \frac{1}{4\pi}x = -\frac{2}{\pi}\left(x^2 - \frac{1}{8}x + \frac{1}{256} - \frac{1}{256}\right)$

$= -\frac{2}{\pi}\left(x - \frac{1}{16}\right)^2 - \frac{1}{28\pi}$ So max is when $x = \frac{1}{16}$ miles

and $r = \frac{1 - 8\left(\frac{1}{16}\right)}{8\pi} = \frac{1 - \frac{1}{2}}{8\pi} = \frac{1}{16\pi}$ miles.

35.



Cost $= 12(2l) + 8(2w) = 24l + 16w = \$4800$

$\Rightarrow w = \frac{600 - 3l}{2}$

$A = lw = l\left(\frac{600-3l}{2}\right) = -\frac{3}{2}l^2 + 300l = A(l)$

$A(l) = -\frac{3}{2}\left(l^2 - 200l + 100^2 - 100^2\right) = -\frac{3}{2}(l - 100)^2 + 15{,}000$

So when $l = 100$ and $w = \frac{600 - 3(100)}{2} = 150$ the area is 15,000

So dimensions of 100yd by 150yd give area of 15,000 yd^2

45. a) Circle $2\pi r = x$ so $r = \frac{x}{2\pi}$ $A = \pi r^2 = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$

Square $4s = 16 - x \Rightarrow s = \frac{16-x}{4}$ $A = s^2 = \left(\frac{16-x}{4}\right)^2 =$

$\frac{256 - 32x + x^2}{16} = 16 - 2x + \frac{1}{16}x^2$

Combined area $A(x) = \frac{x^2}{4\pi} + 16 - 2x + \frac{1}{16}x^2$ D: $(0, \infty)$

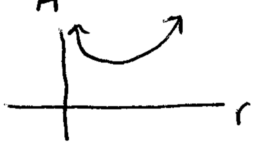
b) this is a parabola pointing up $x = \frac{2}{2 \cdot \frac{4+\pi}{16\pi}} = \frac{16\pi}{4+\pi}$

c) ratio $\frac{\frac{16\pi}{4+\pi}}{16 - \frac{16\pi}{4+\pi}} \cdot \frac{4+\pi}{4+\pi} = \frac{16\pi}{64 + 16\pi - 16\pi} = \frac{16\pi}{64} = \frac{\pi}{4}$

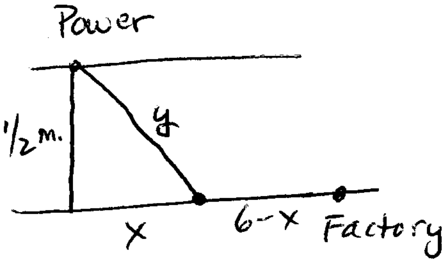
47. a) r is radius of base and h is height $\pi r^2 h = 500$

So $h = \dfrac{500}{\pi r^2}$ $\quad SA = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r\left(\dfrac{500}{\pi r^2}\right)$

$= \pi r^2 + \dfrac{1000}{r}$

b. 

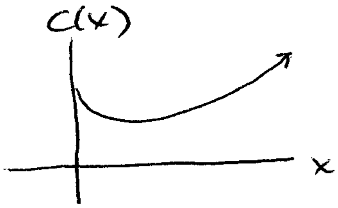
minimum value is ≈ 276.79 cm^2 when $r \approx 5.42$ cm and $h \approx 5.42$ cm

48. 

by Pythagorean Theorem

$\left(\frac{1}{2}\right)^2 + x^2 = y^2 \Rightarrow y = \sqrt{\frac{1}{4} + x^2}$

$C = 5280\left[8y^2 + 6(6-x)\right] = \left[8\sqrt{\frac{1}{4}+x^2} + 36 - 6x\right] \cdot 5280$

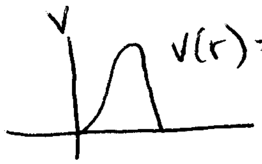
b. Graph 

minimum value is $\approx \$204{,}072$ when run $\approx .57$ miles overland.

5280 feet in a mile so cost per mile across river is \$42,240/mile and overland is \$31,680/mile

50. a) perimeter is $36 = 2r + 2h \Rightarrow r + h = 18 \Rightarrow h = 18 - r$

Volume $= \pi r^2 h = \pi r^2(18 - r) = 18\pi r^2 - \pi r^3$

b) 

So max volume is 2714 unit^3 when radius is 12 units and height is 6 units.

56. $P = 2x + 2r + \frac{1}{2}(2\pi r) = 2x + 2r + \pi r \Rightarrow x = \dfrac{P - 2r - \pi r}{2}$

$A = \frac{1}{2}(\pi r^2) + x(2r) = \dfrac{\pi r^2}{2} + \left(\dfrac{P - 2\pi - \pi r}{2}\right)2r = \dfrac{\pi r^2}{2} + Pr - 2r^2 - \pi r^2$

$= \left(-2 - \frac{\pi}{2}\right)r^2 + Pr = \left(\frac{-4-\pi}{2}\right)r^2 + Pr$ Since $r = \dfrac{-b}{2a} = \dfrac{-P}{-4-\pi} = \dfrac{P}{4+\pi}$

max area is $\dfrac{P^2}{2(4+\pi)}$