

USING TRIG FUNCTIONS TO MODEL DAYLIGHT HOURS

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INTRODUCTION

The hours of daylight on a particular day of the year is a piece of data that repeats every year so should be able to modeled by a trigonometric function. We use a function of the form

$$H = A \sin(Bt - C) + D$$

where t is the day of the year and H is the number of daylight hours.

1. MODELING DAYLIGHT HOURS FOR BOSTON

We have a data set for Boston that was provided to us. We find values for the parameters A, B, C, D as follows:

- (1) The amplitude is

$$A = \frac{M - m}{2} = \frac{15.3 - 9.1}{2} = 3.1.$$

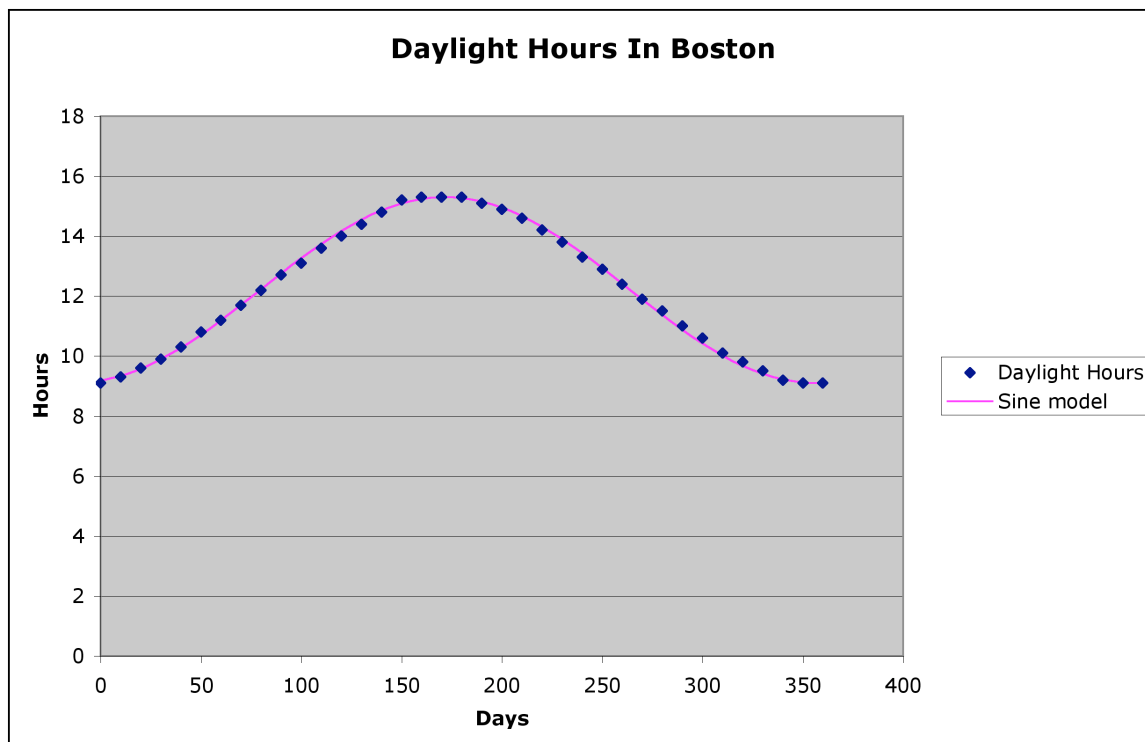
- (2) The period of our data is 365 days. Thus $365 = 2\pi/B$ and $B = 2\pi/365 \approx 0.017$.

- (3) If we had a sine graph with period 365 days then it would have its maximum at $t = 1/4 \times 365 = 91.25$. But the maximum of our data occurs at $t = 170$ so our phase shift needs to be $170 - 91.25 = 78.75$. Then $C/B = 78.75$ so $C = 78.75 \times 2\pi/365 \approx 1.35$.

- (4) So far we have $H = 3.1 \sin(0.017t - 1.35) + D$. Substituting in $t = 0, H = 9.1$ allows us to see that $D = 9.1 - 3.1 \sin(-1.35) \approx 12.12$. This is close to 12 hours which we would expect the average number of hours of sunlight to be.

So our final sine model is

$$H = 3.1 \sin(0.017t - 1.35) + 12.12.$$



Eyeballing the fit to the data shows that this is a reasonable approximation.

2. MODELING DAYLIGHT HOURS FOR FAIRBANKS

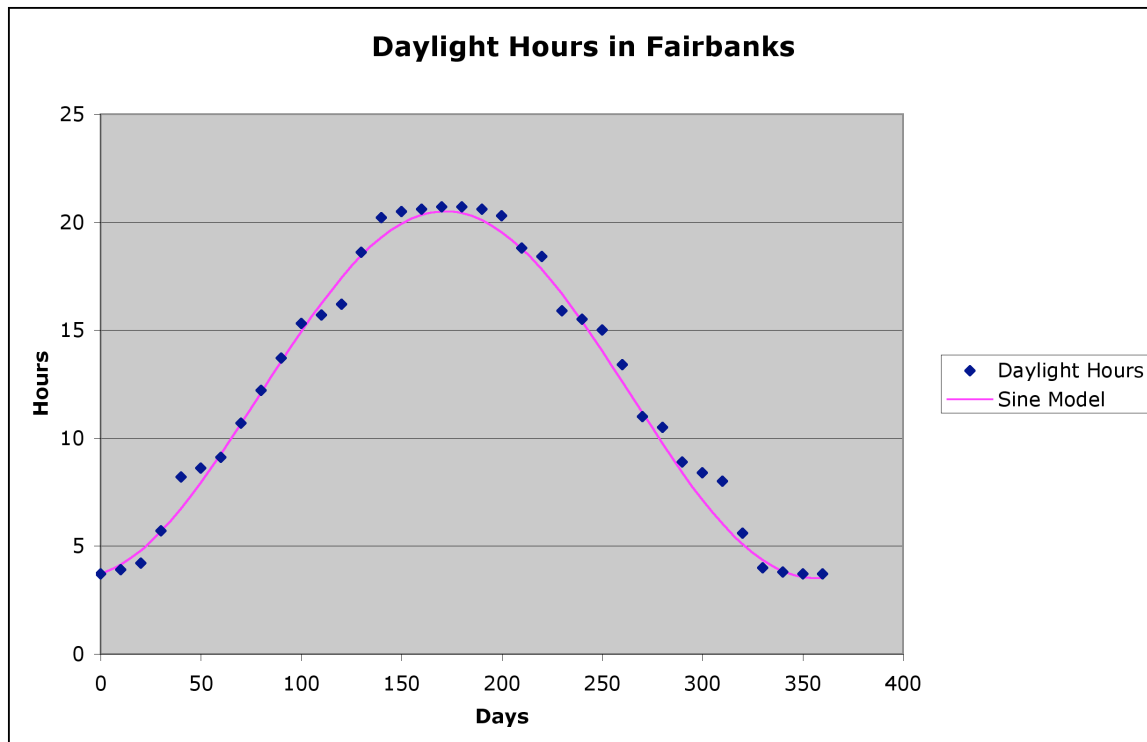
We carry out the same procedure for the next data set. We get the same values for B and C but we have

$$A = \frac{20.7 - 3.7}{2} = 8.5$$

and substituting in the point $t = 0, H = 3.7$ gives $D = 12$.

So our final sine model is

$$H = 8.5 \sin(0.017t - 1.35) + 12.$$



Once again our model seems pretty good, although this data seems slightly less amenable to being modeled by a smooth curve.

3. DISCUSSION OF THE EFFECTS OF LATITUDE

From our two examples above we can conclude some facts about the effect of latitude on the parameters:

- (1) B will always be unchanged.
- (2) C will remain unchanged if we are modeling northern hemisphere data. Otherwise we need to phase shift by a further 6 months. So need $C/B = 78.75 + 182.5$ which gives $C = 4.50$ in the southern hemisphere.
- (3) D should be approximately 12 hours but may need to be adjusted.
- (4) A increases as the distance from the equator increases. At the equator we would expect $A = 0$ and above the Arctic circle ($66^\circ N$) we would expect $A = 12$ as there are complete days of light and complete days of dark. In between, we have the two data points of Boston ($42^\circ N$, $A = 3.1$) and Fairbanks ($64^\circ N$, $A = 8.5$).

So we would have the following approximate equations for daylight hours:

- (1) Tromso ($69^\circ N$): $H = 12 \sin(0.017t - 1.35) + 12$,
- (2) Singapore ($1^\circ N$): $H = 12$,
- (3) Wellington ($42^\circ S$): $H = 3.1 \sin(0.017t - 4.50) + 12$.

4. THE RATE OF DAY LENGTHENING

From the graph for Boston it appears that the greatest rate of increase of daylight hours occurs between $t = 100$ and $t = 110$. This point of maximum rate of increase is an *inflexion point*, where the graph goes from being concave up to concave down. The rate is the slope

$$\frac{13.6 - 13.1}{110 - 100} = 0.05 \text{ hours/day,}$$

which is equal to 3 minutes/day or 21 minutes/week.

The greatest rate of increase for Fairbanks is approximately

$$\frac{15.3 - 13.7}{100 - 90} = 0.16 \text{ hours/day,}$$

which is equal to approximately 10 minutes/day or 70 minutes/week.

The maximum rate of increase is much larger for Fairbanks as it has more variability of day length so the length changes at a higher rate.