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Differential and Complex Geometry of Two-Dimensional Noncommutative Tori

MOMAR DIENG and ALBERT SCHWARZ*
Department of Mathematics, University of California, Davis, CA 95616, U.S.A.
e-mail: {momar, schwarz}@math.ucdavis.edu

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Abstract. We analyze in detail projective modules over two-dimensional noncommutative tori and complex structures on these modules. We concentrate our attention on properties of holomorphic vectors in these modules; the theory of these vectors generalizes the theory of theta-functions. The paper is self-contained; it can be used also as an introduction to the theory of noncommutative spaces with simplest space of this kind thoroughly analyzed as a basic example.

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1. Introduction

Differential geometry of noncommutative spaces, in particular, of noncommutative tori was developed by A. Connes [1, 2]. More detailed analysis of differential geometry of noncommutative tori was performed in [3, 4, 6, 14–17].

The interest in this subject was partly motivated by the applications to string/M-theory found in [4] (see [7] for a review). Complex geometry of noncommutative tori and of projective modules over them was studied in [18] in connection with noncommutative generalization of theta-functions.

The goal of this Letter is to illustrate the general results of [18] using the example of two-dimensional noncommutative tori, where all calculations can be performed explicitly. We repeat all the basic definitions, starting with the definition of a noncommutative torus. Therefore, the paper can be read independently of [18] and of other papers about noncommutative tori. However, in the rest of the introduction, we assume some knowledge of preceding work.

In Section 2 we describe projective modules over a two-dimensional noncommutative torus and their tensor products. The results of this section are closely related to some results proved in multidimensional case in [18]. We will formulate these general results, restricting ourselves to basic T_θ-modules (modules with a constant curvature connection where the endomorphism algebra is another noncommutative

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torus $T_{\theta'}$). Instead of basic modules, we can work with corresponding $(T_{\theta'}, T_\theta)$-bimodules. Tensor product of $(T_{\theta'}, T_\theta)$-bimodule $\mathcal{E}_1$ and $(T_\theta, T_{\theta''})$-bimodule $\mathcal{E}_2$ is a $(T_{\theta'}, T_{\theta''})$-bimodule $\mathcal{E}$. We can find the bimodule $\mathcal{E}$ using the fact that basic bimodules $\mathcal{E}_1$ and $\mathcal{E}_2$ can be described by means of elements $g_1, g_2 \in SO(d, d, \mathbb{Z})$ obeying $\theta' = g_1\theta, \theta'' = g_2\theta$. Here g_1, g_2 act on the parameter of noncommutativity – antisymmetric matrix θ – by means of fractional linear transformation. The description of $\mathcal{E}$ follows from the relation $\theta'' = g_2 g_1^{-1}\theta'$. In the two-dimensional case, all basic right T_θ-modules can be represented as modules $E_{n,m}$ (see Section 2) where m and n are relatively prime. The space $E_{n,m}$ can be considered also as a left $T_{\theta'}$-module $E'_{a,m}$ where

$$an - bm = 1, \quad a, b \in \mathbb{Z}, \qquad \theta' = (a\theta + b)(m\theta + n)^{-1},$$

or as a $(T_{\theta'}, T_\theta)$-bimodule that will be denoted by $\mathcal{E}_1$. One can use the general results to calculate the tensor product $\mathcal{E}_1 \otimes_{T_\theta} \mathcal{E}_2$ where $\mathcal{E}_2$ is a $(T_\theta, T_{\theta''})$-bimodule, corresponding to the basic left T_θ-module $E'_{k,l}$. It follows from these results that this tensor product considered as a left $T_{\theta'}$-module is isomorphic to $E'_{ak+bl,nl+mk}$. We give a direct proof of this relation constructing an explicit isomorphism. This construction is used essentially in Section 3, devoted to complex geometry of projective modules over two-dimensional noncommutative tori. We calculate holomorphic vectors (theta-vectors) in basic modules and tensor products of these vectors. The relations we obtain generalize some well-known relations for theta-functions. The appendix to the ArXiv version of the paper (math.QA/0203160) contains a very detailed calculation of the tensor product (we did not include the appendix into the journal version). We did not try to analyze the connection of our results to Manin's version of the theory of noncommutative theta-functions ([8–10]). This is an interesting problem.

2. Projective Modules and their Tensor Products

One can define the algebra T_θ^d of smooth functions on d-dimensional noncommutative torus as an algebra of formal linear combinations $f = \sum C_\mathbf{n} U_\mathbf{n}$ where $\mathbf{n} \in \mathbb{Z}^d$ and $C_\mathbf{n}$ are complex numbers tending to zero at infinity faster then any power and the multiplication is governed by the rule

$$U_\mathbf{n} U_\mathbf{m} = e^{\pi i \mathbf{n}\theta\mathbf{m}} U_{\mathbf{n}+\mathbf{m}},$$

where $\theta = \theta^{\alpha\beta}$ is a real antisymmetric matrix. An antilinear involution on T_θ^d is defined by the requirement $U_\mathbf{n}^* = U_{-\mathbf{n}}$. These operations together with the standard structure of vector space permit us to consider T_θ^d as involutive associative algebra with unit element $1 = U_\mathbf{0}$ and with trace $\mathrm{Tr} f = C_\mathbf{0}$. We will fix our attention on the case $d = 2$; in this case the matrix $\theta^{\alpha\beta}$ can be specified by means of one number $\theta^{12} = \theta$ and the multiplication is specified by the relation $U_1 U_2 = e^{2\pi i\theta} U_2 U_1$, where $U_\alpha = U_{\mathbf{e}_\alpha}$ are elements $U_\mathbf{n}$ corresponding to vectors of standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$.

In the case when θ is a natural number T_θ^2 is isomorphic to the algebra of smooth functions on the commutative two-dimensional torus. We will consider projective modules over T_θ^2 (we always assume that our modules are finitely generated; a projective module is by definition a direct summand in a free module $(T_\theta^2)^k$). We assume that the number θ is irrational; in this case every projective right module over T_θ^2 is isomorphic to one of the modules $E_{n,m}$ defined in the following way (see [2, 3, 7]). The elements of $E_{n,m}$ are functions on the Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}_m)$ and the action of the generators U_1, U_2 of T_θ^2 is given by:

$$U_1 f(x, \mu) = f\left(x - \frac{n + m\theta}{m}, \mu - 1\right),$$

$$U_2 f(x, \mu) = e^{2\pi i(x - \mu n/m)} f(x, \mu)$$

We can define also left modules over T_θ^2 replacing θ by $-\theta$ in the above formulas; we will use the notation $E'_{n,m}$ for left modules. If it is necessary to emphasize that we consider $E_{n,m}$ or $E'_{n,m}$ as a T_θ-modules we will use the notation $E_{n,m}(\theta)$ or $E'_{n,m}(\theta)$. We will assume that the numbers m and n are relatively prime. Corresponding modules $E_{n,m}$ are called basic modules. Every module over T_θ^2 can be represented as a direct sum of isomorphic basic modules. In this case the algebra of endomorphisms of the module $E = E_{n,m}$ is again a noncommutative torus $T_{\theta'}$ generated by operators

$$Z_1 f(x, \mu) = f\left(x - \frac{1}{m}, \mu - a\right),$$

$$Z_2 f(x, \mu) = \exp\left[2\pi i\left(\frac{x}{n + m\theta} - \frac{\mu}{m}\right)\right] f(x, \mu),$$

where a, b are integers obeying $an - bm = 1$. This fact follows from relation $Z_1 Z_2 = e^{-2\pi i\theta'} Z_2 Z_1$, where $\theta' = (b + a\theta)/(n + m\theta)$. One can consider E as a left $T_{\theta'}$-module; this module is isomorphic to $E'_{a,m}$.

We can regard E also as $(T_{\theta'}, T_\theta)$-bimodule, since the action of $T_{\theta'}$ commutes with the action of T_θ. Notice that the bimodule E depends on the choice of a and b. The tori T_θ and $T_{\theta'}$ are Morita equivalent and the bimodule E is a Morita equivalence bimodule ([2, 3, 7]).

Recall that two associative algebras A and B are Morita equivalent if corresponding categories of modules are equivalent. Having an (A, B)-bimodule P we can assign to every right A-module E a right B-module $E' = E \otimes_A P$ where tensor product over A is obtained from the standard tensor product over $\mathbb{C}$ by means of identification $ea \otimes p \sim e \otimes ap$. If this correspondence is invertible, we say that P is a Morita equivalence bimodule.

Let us now calculate the tensor product $E_{n,m} \otimes_{T_\theta} E'_{k,l}$ where $E_{n,m}$ is a right T_θ^2-module and $E'_{k,l}$ is a left T_θ^2-module. We will see that this product can be considered a vector space $\mathcal{E} = \mathcal{S}(\mathbb{R} \times \mathbb{Z}_{nl+mk})$. Namely, the formula

$$h(z,\Delta)=\sum_{q\in\mathbb{Z}}\Bigg[f\bigg((n+m\theta)z-\frac{n+m\theta}{m}q+\frac{l(n+m\theta)}{m(nl+mk)}\Delta,-q+a\Delta\bigg)\times$$
$$\times\, g\bigg((n+m\theta)z+\frac{k-l\theta}{l}q-\frac{k-l\theta}{nl+mk}\Delta,q\bigg)\Bigg],\tag{1}$$

where $0\leqslant\Delta\leqslant nl+mk-1$, specifies a map of tensor product over $E_{n,m}\otimes_{T_\theta}E'_{k,l}$ onto $\mathcal{E}$, so that, with the usual notation, $f\in E_{n,m}(\theta)$, $g\in E'_{k,l}(\theta)$ and $h\in E'_{ak+bl,nl+mk}(\theta')$. This map is an isomorphism between the tensor product and $\mathcal{E}$. The formula (1) determines a bilinear map that is compatible with identification $ea\otimes p\sim e\otimes ap$ (see the Appendix to the ArXiv version for details). We can consider $E_{n,m}$ as a $(T_{\theta'},T_\theta)$-bimodule, where

$$\theta'=\frac{b+a\theta}{n+m\theta},\quad an-bm=1$$

and $E_{k,l}$ as a $(T_\theta,T_{\theta''})$-bimodule where

$$\theta''=-\frac{d-c\theta}{k-l\theta},\quad ck-dl=1.$$

Then $\mathcal{E}$ can be regarded as $(T_{\theta'},T_{\theta''})$-bimodule. It follows from (1) that $\mathcal{E}$ considered as a left $T_{\theta'}$-module is isomorphic to $E'_{N',M}$ where $M=nl+mk$, $N'=ak+bl$ and $\mathcal{E}$ considered as a right $T_{\theta''}$-module is isomorphic to $E_{N'',M}$, where $N''=-(cn+md)$ for $ck-dl=1$. This result can be obtained also from general considerations (see the Introduction).

Let us apply (1) to the special case when

$$f(x,\mu)=\mathrm{e}^{-\frac{1}{2}\sigma_1x^2-c_1x}\delta^\mu_\alpha,$$

$$g(y,v)=\mathrm{e}^{-\frac{1}{2}\sigma_2y^2-c_2x}\delta^v_\beta,$$

where

$$\sigma_1=\mathrm{i}\tau_1\frac{m}{n+m\theta},\quad \mu\in\mathbb{Z}_m,$$

$$\sigma_2=\mathrm{i}\tau_2\frac{l}{k-l\theta},\quad v\in\mathbb{Z}_l$$

and δ^i_j is the usual Kronecker delta. It is straightforward to check that in this case:

$$h_{\alpha\beta}(z,\Delta)=\sum_{q\in\mathbb{Z}}\Bigg[\exp\bigg\{\frac{-\sigma_1}{2}\bigg((n+m\theta)z-\frac{n+m\theta}{m}q+\frac{l(n+m\theta)}{m(nl+mk)}\Delta\bigg)^2-$$
$$-c_1\bigg((n+m\theta)z-\frac{n+m\theta}{m}q+\frac{l(n+m\theta)}{m(nl+mk)}\Delta\bigg)\bigg\}\delta_\alpha^{-q-a\Delta}\times$$
$$\times\exp\bigg\{-\frac{\sigma_2}{2}\bigg((n+m\theta)z+\frac{k-l\theta}{l}q-\frac{k-l\theta}{nl+mk}\Delta\bigg)^2-$$
$$-c_2\bigg((n+m\theta)z+\frac{k-l\theta}{l}q-\frac{k-l\theta}{nl+mk}\Delta\bigg)\bigg\}\delta^q_\beta\Bigg]\tag{2}$$

Notice that the right-hand side of (2) can be expressed in terms of theta-functions; first note that the set of solutions to the system of congruences:

$$q = a\Delta - \alpha \pmod{m}, \qquad q = \beta \pmod{l}$$

can be written as $q_o + u(ml/r)$ for some integer q_o, $r = \gcd(m, l)$, and $u \in \mathbb{Z}$. Hence, we can substitute $q_o + u(ml/r)$ in for q in (2), do away with Kronecker deltas, and sum over u instead of q. We obtain:

$$h_{\alpha\beta}(z, \Delta) = \Theta(s, t) \cdot \xi(z, \Delta),$$

where

$$\xi_{\alpha\beta}(z, \Delta) = \exp\Bigg[-\frac{\sigma_1}{2}\left((n+m\theta)z + \frac{l(n+m\theta)\Delta}{m(nl+mk)}\right)^2 - \frac{\sigma_2}{2}\left((n+m\theta)z - \frac{(k-l\theta)\Delta}{nl+mk}\right)^2 - \\ - (c_1+c_2)(n+m\theta)z - \left(c_1\frac{l(n+m\theta)}{m(nl+mk)} - c_2\frac{k-l\theta}{nl+mk}\right)\Delta - \\ - \left(\sigma_1\left(\frac{n+m\theta}{m}\right)^2 + \sigma_2\left(\frac{k-l\theta}{l}\right)^2\right)\frac{q_o^2}{2}\Bigg],$$

$$s = -\frac{\sigma_1 l^2(n+m\theta)^2 + \sigma_2 m^2(k-l\theta)^2}{2\pi i r^2},$$

$$t = \frac{\sigma_1(n+m\theta)^2}{2\pi i m}\left(z + \frac{l\Delta}{m(nl+mk)} - \frac{lq_o}{r}\right) + c_1\frac{n+m\theta}{2\pi i m} - c_2\frac{k-l\theta}{2\pi i l} + \\ + \frac{\sigma_2(k-l\theta)^2}{2\pi i l}\left(-\frac{n+m\theta}{k-l\theta}z + \frac{\Delta}{nl+mk} - \frac{mq_o}{r}\right)$$

and $\Theta(s, t)$ a classical theta-function. We recall that classical-theta functions are defined as

$$\Theta(s, t) = \sum_{u\in\mathbb{Z}} e^{\pi i s u^2 + 2\pi i t u}, \quad \operatorname{Im} s > 0$$

3. Connections and Complex Structures

One can define a connection on a right T_θ^d-module E as a set of $\mathbb{C}$-linear operators $\nabla_1, \ldots, \nabla_d$ obeying the Leibniz rule:

$$\nabla_\alpha(ea) = \nabla_\alpha e \cdot a + e \cdot \delta_\alpha a, \tag{3}$$

where $e \in E, a \in T_\theta^d$ and the derivatives $\delta_\alpha a$ are specified by the formula $\delta_\alpha U_{\mathbf{n}} = 2\pi n i U_{\mathbf{n}}$ (see [1]). It will be convenient for us to generalize the notion of connection replacing δ_α in (3) by $\delta'_\alpha = a_\alpha^\beta \delta_\beta$ where a_α^β is a nondegenerate matrix. A connection $\nabla_1, \ldots, \nabla_d$ is a constant curvature connection if

$$[\nabla_\alpha, \nabla_\beta] = i f_{\alpha\beta} \cdot 1,$$

where $f_{\alpha\beta}$ are numbers and 1 stands for identity operator. Similar definitions can be given for left modules. We always consider unitary connections (i.e. operators ∇_α should be anti Hermitian). It is easy to check that the operators

$$\nabla_1 = 2\pi \mathrm{i} \frac{m}{n + m\theta} x, \quad \nabla_2 = 2\pi \frac{\mathrm{d}}{\mathrm{d}x}. \tag{4}$$

specify a constant curvature connection of right T_θ-module $E_{m,n}$. The same operators determine a connection (in a generalized sense) on $E_{n,m}$ considered as a left $T_{\theta'}$-module. This follows from relations $[\nabla_1, \nabla_2] = (2\pi \mathrm{i} m)/(n + m\theta)$. We see that ∇_1, ∇_2 can be considered as a constant curvature connection on a $(T_{\theta'}, T_\theta)$-bimodule.

If ∇_1, ∇_2 is a constant curvature connection on a module E, we can introduce a complex structure on E fixing $\bar{\partial}$-connection $\bar{\nabla} = \lambda_1 \nabla_1 + \lambda_2 \nabla_2$, where λ_1 and λ_2 are complex numbers and the quotient $\tau = \lambda_1/\lambda_2$ is not real [18]. This complex structure on a T_θ-module corresponds to the complex structure on T_θ, specified by means of $\bar{\partial}$-derivative $\lambda_1\delta_1 + \lambda_2\delta_2$. By definition, vector $\Theta \in E$ is holomorphic if $\bar{\nabla}\Theta = 0$. Notice that the notion of holomorphicity depends only on $\tau = \lambda_1/\lambda_2$, therefore we say that $\bar{\nabla}$ and $\rho\bar{\nabla}$ where $\rho \neq 0$ determine the same complex structure on E. Holomorphic vectors are closely related to theta-functions, hence we call holomorphic vectors in basic modules theta vectors.

Let us consider holomorphic vectors in T_θ-modules $E_{n,m}$, assuming that n and m are relatively prime. In this case all constant curvature connections have the form $\nabla_\alpha = \nabla_\alpha^0 + c_\alpha$, where ∇_α^0 stand for the connection (4) and c_1, c_2 are constants. The equation $\bar{\nabla}\Theta = 0$ takes the form

$$\left(\mathrm{i}\tau \frac{m}{n + m\theta} x + \frac{\partial}{\partial x} + c\right) \varphi(x, \mu) = 0.$$

If $\operatorname{Im} \tau < 0$ it has m linearly independent solutions

$$\varphi_\alpha(x, \mu) = \mathrm{e}^{-\frac{1}{2}\sigma x^2 - cx} \delta_\alpha^\mu, \tag{5}$$

where

$$\sigma = \mathrm{i}\tau \frac{m}{n + m\theta}, \quad \mu \in \mathbb{Z}_m.$$

The functions (5) belong to $\mathcal{S}$ only if $\operatorname{Re} \sigma > 0$. We assumed that $n + m\theta > 0$; in the case $n + m\theta < 0$, the condition of existence of holomorphic vectors is that $\operatorname{Im} \tau > 0$. We see that in the case when holomorphic vectors exist, the space $\mathcal{H}_{n,m}$ of holomorphic vectors in $E_{n,m}$ is m-dimensional; the functions (5) constitute a basis of $\mathcal{H}_{n,m}$. Considering $E_{n,m}$ as a $(T_{\theta'}, T_\theta)$-bimodule and taking into account that a constant curvature connection on $E_{n,m}$ is a constant curvature connection on the bimodule, one can define a notion of complex structure and of holomorphic vector for a bimodule. More precisely, complex structure on T_θ-module $E_{n,m}$ induces a complex structure on same space considered as left $T_{\theta'}$-module in such a way that the notion of holomorphic vector remains the same. These two complex structures specify a complex structure on a bimodule; the notion of holomorphic vector in a bimodule coincides with corresponding notion for both modules. If $\mathcal{E}'$ is a complex $(T_{\theta'}, T_\theta)$-bimodule, $\mathcal{E}''$ is a complex $(T_\theta, T_{\theta''})$-bimodule, then the $(T_{\theta'}, T_{\theta''})$-bimodule $\mathcal{E} = \mathcal{E}' \otimes_{T_\theta} \mathcal{E}''$ can be equipped with complex structure. We assume that complex structure on the right T_θ-module $\mathcal{E}'$ and on the left T_θ-module $\mathcal{E}''$ correspond to the same complex structure on T_θ. One can prove [18] that the tensor product of two holomorphic vectors is again a holomorphic vector (i.e. we have a natural map $\mathcal{H}' \otimes \mathcal{H}''$ into $\mathcal{H}$ where $\mathcal{H}', \mathcal{H}''$ and $\mathcal{H}$ stand for spaces of holomorphic vectors in $\mathcal{E}', \mathcal{E}''$ and $\mathcal{E}$ correspondingly). Let the basis of $\mathcal{H}'$ consist of

$$\varphi'_\alpha = \mathrm{e}^{-\frac{1}{2}\sigma_1 x^2 - c_1 x}\delta^\mu_\alpha,$$

where

$$\sigma_1 = \mathrm{i}\tau_1 \frac{m}{n + m\theta}, \qquad \mu \in \mathbb{Z}_m,\ \alpha \in \{1, \ldots, m\}$$

and the basis of $\mathcal{H}''$ consist of

$$\varphi''_\beta = \mathrm{e}^{-\frac{1}{2}\sigma_2 y^2 - c_2 x}\delta^\nu_\beta,$$

where

$$\sigma_2 = \mathrm{i}\tau_2 \frac{l}{k - l\theta}, \qquad \nu \in \mathbb{Z}_l,\ \alpha \in \{1, \ldots, l\}$$

We assume that $\mathcal{E}'$, considered as a right T_θ-module, is isomorphic to $E_{n,m}$, and that $\mathcal{E}''$, considered as a left T_θ-module, is isomorphic to $E'_{k,l}$. We can use (2) to calculate $\varphi'_\alpha \otimes_{T_\theta} \varphi''_\beta$. The condition that the complex structures on $\mathcal{E}'$ and $\mathcal{E}''$ correspond to the same complex structure on $\mathcal{E}$ implies $\tau_1 = \tau_2$. Using this, we can check that the theta-functions that appear in (2) do not depend on z in our case. Applying (2), we obtain that, $\varphi'_\alpha \otimes_{T_\theta} \varphi''_\beta$ maps to $\Xi_{\alpha\beta}(z, \Delta) \in \mathcal{H}$ which is of the form

$$\Xi_{\alpha\beta}(z, \Delta) = \sum_\gamma c^\gamma_{\alpha\beta}\varphi_\gamma(z, \Delta),$$

where

$$\varphi_\gamma(z, \Delta) = \exp\left[\frac{\pi\mathrm{i}\tau(nl + mk)(n + m\theta)}{k - l\theta} z^2 - (c_1 + c_2)(n + m\theta)z\right]\delta^\Delta_\gamma$$

constitute a basis of $\mathcal{H}$ and the constants $c^\gamma_{\alpha\beta}$ are given by $c^\gamma_{\alpha\beta} = \Theta(s, t) \cdot \mathrm{e}^K$, where $\Theta(s, t)$ a classical theta-function,

$$K = -\frac{\pi\mathrm{i}\tau l}{m(nl + mk)}\gamma^2 - \frac{(l(n + m\theta)c_1 - m(k - l\theta)c_2)}{m(nl + mk)}\gamma + \pi\mathrm{i}sq_o^2 + 2\pi\mathrm{i}tq_o$$

and

$$s = -ml(nl + mk),$$

$$t = \frac{l\tau\Delta - q_o(nl + mk)}{r} + \frac{c_1 l(n + m\theta) - c_2 m(k - l\theta)}{2\pi\mathrm{i}r}$$

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