The Geometry of Polygon Spaces

Victor Piercey and Matthew Thomas

December 6, 2007

Moduli Spaces

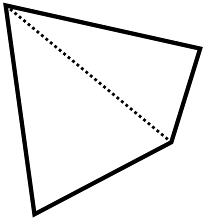
Moduli spaces are parameter spaces of algebraic varieties. Moduli spaces of polygons are examples with a clear geometric meaning.

The moduli space of a given polygon with fixed side lengths is defined as the space of deformations in 3-dimensional Euclidean space.

The trivial case is the moduli space of a triangle. Since a triangle cannot be deformed while keeping side lengths fixed, the moduli space of a triangle is a point.

The General Quadrilateral

A simple example to keep in mind is a generic quadrilateral, shown below. The dotted line represents a diagonal. We refer to the length of this diagonal as the diameter and use the diameter as a parameter for the moduli space.



Given a specific diameter between the minimum and maximum, we may rotate the upper right point around the diameter, allowing a degree of freedom. Graphing the parameter space, we see that $\mathbb{S}^2$ is the parameter space of the quadrilateral shown.

The General Quadrilateral

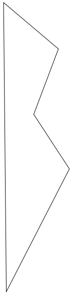
- In general, a polygon which has one side significantly longer than the others will have moduli space $\mathbb{CP}^{n-3}$.

The General Quadrilateral

- In general, a polygon which has one side significantly longer than the others will have moduli space $\mathbb{CP}^{n-3}$.
- To make this precise would require a bit of work, so we will specify this to mean that one side length is equal to the sum of the other side lengths minus a small number, ϵ, as shown.

The General Quadrilateral

- In general, a polygon which has one side significantly longer than the others will have moduli space $\mathbb{CP}^{n-3}$.
- To make this precise would require a bit of work, so we will specify this to mean that one side length is equal to the sum of the other side lengths minus a small number, ϵ, as shown.



The General Quadrilateral

We will consider our points to be on the the Riemann sphere and will specfiy without loss of generality that:

- our first vertex will be placed at ∞

The General Quadrilateral

We will consider our points to be on the the Riemann sphere and will specfiy without loss of generality that:

- our first vertex will be placed at ∞
- the second will be placed at the origin.

The General Quadrilateral

We will consider our points to be on the the Riemann sphere and will specfiy without loss of generality that:

- our first vertex will be placed at ∞
- the second will be placed at the origin.

This will require our first side to lie along the z-axis.

The General Quadrilateral

- Each side is a vector in $\mathbb{R}^3$ that can be translated to the origin and normalized.
- Each side corresponds to a point on the Riemann sphere.
- We specify without loss of generality that our first such point is located at ∞ (the North pole) and the second at the origin (the South pole).
- Require that our first side length is the longest side.
- As a consequence, in order for our polygon to be closed, no other side can be parallel to the first side.

Overview

- Symplectic Geometry
- Geometric Invariant Theory
- The Kirwan-Kempf-Ness Theorem
- Polygon Spaces
- Stable Polygons/Curves
- Results

Symplectic Forms

- A *symplectic form* is a nondegenderate closed 2-form.

Symplectic Forms

- A *symplectic form* is a nondegenerate closed 2-form.
- A symplectic manifold is a manifold M with a symplectic form ω such that for each $p \in M$, $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is skew-symmetric and bilinear.

A useful example here is $\mathbb{S}^2$ in $\mathbb{R}^3$.

- The symplectic form is induced by the map

$$\omega_p(u, v) := < p, u \times v > .$$

- This is a top degree form, hence closed.
- It is nondegenerate because if $u \neq 0$, then for $v = u \times p$, $< p, u \times v > \neq 0$.

The Moment Map

- Let (M, ω) be a symplectic manifold
- G be a Lie group acting symplectically on (M, ω) (This means that the group action preserves the symplectic form ω.)
- $\mathfrak{g}$ be its Lie algebra, and
- $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$.

The Moment Map

- Let (M, ω) be a symplectic manifold
- G be a Lie group acting symplectically on (M, ω) (This means that the group action preserves the symplectic form ω.)
- $\mathfrak{g}$ be its Lie algebra, and
- $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$.
- In our case, it will suffice to think of the Lie group and its Lie algebra as matrix spaces.

The Moment Map

For each $a \in \mathfrak{g}$, there is an associated vector field on (M, ω) which we will denote by $x \mapsto a_x$.

The Moment Map

For each $a \in \mathfrak{g}$, there is an associated vector field on (M, ω) which we will denote by $x \mapsto a_x$.

- The moment map for the action of G on (M, ω) is a map $\mu : (M, \omega) \to \mathfrak{g}^*$ which is G equivariant with respect to the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$ and satisfies:

$$\langle d\mu(x)(\xi), a\rangle = \omega_x(\xi, a_x) \ \forall x \in M, \xi \in T_x M.$$

The Moment Map

For each $a \in \mathfrak{g}$, there is an associated vector field on (M, ω) which we will denote by $x \mapsto a_x$.

- The moment map for the action of G on (M, ω) is a map $\mu : (M, \omega) \to \mathfrak{g}^*$ which is G equivariant with respect to the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$ and satisfies:

$$\langle d\mu(x)(\xi), a \rangle = \omega_x(\xi, a_x) \ \forall x \in M, \xi \in T_x M.$$

- As an example, consider the action of $\mathrm{SO}(3)$ on $\mathbb{S}^2$. The moment map is the inclusion of $\mathbb{S}^2$ into $\mathbb{R}^3$.

An Example of the Moment Map

- An application: moment maps tell us that angular momentum is preserved in a mechanical system.
- Note that not all actions admit moment maps.

An Example of the Moment Map

- An application: moment maps tell us that angular momentum is preserved in a mechanical system.
- Note that not all actions admit moment maps.
- Actions that do admit moment maps are called Hamiltonian actions.

Symplectic Reduction

- Consider a moment map:

$$\Phi_\omega : X \to k^*.$$

Symplectic Reduction

- Consider a moment map:

$$\Phi_\omega : X \to k^*.$$

- k is the Lie algebra of K, a compact Lie group which acts symplectically on X.

Symplectic Reduction

- Consider a moment map:

$$\Phi_\omega : X \to k^*.$$

- k is the Lie algebra of K, a compact Lie group which acts symplectically on X.
- K acts freely on $\Phi_\omega^{-1}(0)$.

Symplectic Reduction

- Consider a moment map:

$$\Phi_\omega : X \to k^*.$$

- k is the Lie algebra of K, a compact Lie group which acts symplectically on X.
- K acts freely on $\Phi_\omega^{-1}(0)$.
- The orbit space $\Phi_\omega^{-1}(0)/K$ is naturally a symplectic manifold by the Marsden-Weinstein-Meyer theorem

- An n-gon will be determined by a set of n points in 3-dimensional Euclidean space, $v_1, v_2, ..., v_n$ which form the vertices of the polygon.
- The edges $e_1, e_2, ..., e_n$ will be formed be cyclically connecting the vertices. (For $1 \leq i \leq n-1$, e_i connects v_i and v_{i+1} and e_n connects v_n and v_1.)

Basic Definitions

- If we consider the edges to be vectors in $\mathbb{R}^3$, then we require

$$\sum_{i=1}^{n} e_i = 0$$

so that the polygon is closed.

Basic Definitions

- If we consider the edges to be vectors in $\mathbb{R}^3$, then we require

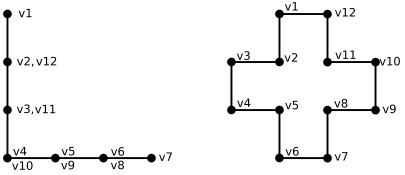
$$\sum_{i=1}^{n} e_i = 0$$

so that the polygon is closed.

- We identify polygons which can be obtained from each other by orientation-preserving isometries of $\mathbb{E}^3$.
- To do so, we first translate so that v_1 lies at the origin.
- The remaining idendtifications come from the natural action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$.

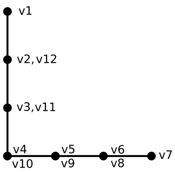
An Example With 12-gons

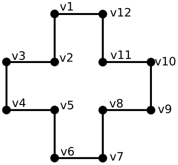
The following diagrams illustrate possible 12-gons:



An Example With 12-gons

The following diagrams illustrate possible 12-gons:

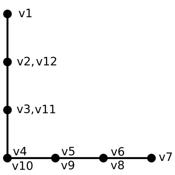


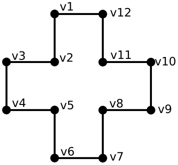


- side lengths are said to be *parallel* if they point in exactly the same direction and
- *anti-parallel* if they point in opposite directions.

An Example With 12-gons

The following diagrams illustrate possible 12-gons:

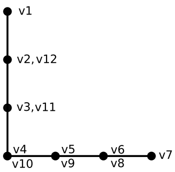


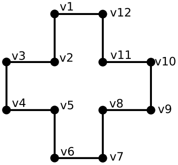


- side lengths are said to be *parallel* if they point in exactly the same direction and
- *anti-parallel* if they point in opposite directions.
- We can assume that parallel edges are adjacent, since we can always permute the edges and use the inverse of the permutation to return to where we started.

An Example With 12-gons

The following diagrams illustrate possible 12-gons:





- side lengths are said to be *parallel* if they point in exactly the same direction and
- *anti-parallel* if they point in opposite directions.
- We can assume that parallel edges are adjacent, since we can always permute the edges and use the inverse of the permutation to return to where we started.
- We will also say that a set of edges is *degenerate* if that set of edges is a maximal set which is parallel in the polygon.

The Moduli Space M_r

- $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$ is an n-tuple of positive real numbers

The Moduli Space M_r

- $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$ is an n-tuple of positive real numbers
- define M_r to be the space of n-gons with side lengths $r_1, \ldots, r_n$ modulo $\mathrm{SO}(3)$.

The Moduli Space M_r

- $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ is an n-tuple of positive real numbers
- define M_r to be the space of n-gons with side lengths $r_1, \ldots, r_n$ modulo $\mathrm{SO}(3)$.
- We can always rescale r since $\forall \lambda \in \mathbb{R}_+$, there is an isomorphism $M_r \cong M_{\lambda r}$.

The Moduli Space M_r

- $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ is an n-tuple of positive real numbers
- define M_r to be the space of n-gons with side lengths $r_1, \ldots, r_n$ modulo $\mathrm{SO}(3)$.
- We can always rescale r since $\forall \lambda \in \mathbb{R}_+$, there is an isomorphism $M_r \cong M_{\lambda r}$.
- We make the choice to normalize r so that

$$\sum_{i=1}^{n} r_i = 2.$$

The Moduli Space M_r

- set $\mathcal{P}_n$ to be the space of all n-gons

The Moduli Space M_r

- set $\mathcal{P}_n$ to be the space of all n-gons
- define $\pi : \mathcal{P} \to \mathbb{R}^n_+$ by assigning an n-gon P the vector whose components correspond to the lengths of the edges of P

The Moduli Space M_r

- set $\mathcal{P}_n$ to be the space of all n-gons
- define $\pi : \mathcal{P} \to \mathbb{R}^n_+$ by assigning an n-gon P the vector whose components correspond to the lengths of the edges of P
- The image of π lies in the hypersimplex

$$\mathbb{D}^n_2 = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1, \sum_i x_i = 2 \right\}.$$

The Moduli Space M_r

- set $\mathcal{P}_n$ to be the space of all n-gons
- define $\pi : \mathcal{P} \to \mathbb{R}^n_+$ by assigning an n-gon P the vector whose components correspond to the lengths of the edges of P
- The image of π lies in the hypersimplex

$$\mathbb{D}^n_2 = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1, \sum_i x_i = 2 \right\}.$$

- M_r is the fiber $\pi^{-1}(r)$ for $r \in \mathbb{D}^n_2$.

The Moduli Space M_r

- set $\mathcal{P}_n$ to be the space of all n-gons
- define $\pi : \mathcal{P} \to \mathbb{R}^n_+$ by assigning an n-gon P the vector whose components correspond to the lengths of the edges of P
- The image of π lies in the hypersimplex

$$\mathbb{D}^n_2 = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1, \sum_i x_i = 2 \right\}.$$

- M_r is the fiber $\pi^{-1}(r)$ for $r \in \mathbb{D}^n_2$.
- We assume for the rest of the talk that the side-length vector r is in the interior of the hypersimplex $\mathbb{D}^n_2$.

A Complex Analytic Structure on M_r

- We now have the tools to define a complex analytic structure on M_r.

A Complex Analytic Structure on M_r

- We now have the tools to define a complex analytic structure on M_r.
- Define the following subspace of $(\mathbb{S}^2)^n$:

$$\tilde{\mathcal{M}}_r = \left\{ \vec{u} \in (\mathbb{S}^2)^n : \sum_{i=1}^{n} r_i u_i = 0 \right\}.$$

A Complex Analytic Structure on M_r

- We now have the tools to define a complex analytic structure on M_r.
- Define the following subspace of $(\mathbb{S}^2)^n$:

$$\tilde{\mathcal{M}}_r = \left\{ \vec{u} \in (\mathbb{S}^2)^n : \sum_{i=1}^{n} r_i u_i = 0 \right\}.$$

- Define $\mathcal{M}_r = \tilde{\mathcal{M}}_r / \mathrm{SO}(3)$.

A Complex Analytic Structure on M_r

- We now have the tools to define a complex analytic structure on M_r.
- Define the following subspace of $(\mathbb{S}^2)^n$:

$$\tilde{\mathcal{M}}_r = \left\{ \vec{u} \in (\mathbb{S}^2)^n : \sum_{i=1}^{n} r_i u_i = 0 \right\}.$$

- Define $\mathcal{M}_r = \tilde{\mathcal{M}}_r / \mathrm{SO}(3)$.
- If $P \in M_r$, normalizing the edges gives a collection $u_i = e_i / r_i \in \mathbb{S}^2$.

A Complex Analytic Structure on M_r

- We now have the tools to define a complex analytic structure on M_r.
- Define the following subspace of $(\mathbb{S}^2)^n$:

$$\tilde{\mathcal{M}}_r = \left\{ \vec{u} \in (\mathbb{S}^2)^n : \sum_{i=1}^{n} r_i u_i = 0 \right\}.$$

- Define $\mathcal{M}_r = \tilde{\mathcal{M}}_r / \mathrm{SO}(3)$.
- If $P \in M_r$, normalizing the edges gives a collection $u_i = e_i / r_i \in \mathbb{S}^2$.
- The closing condition for P to be a polygon requires

$$\sum_{i=1}^{n} r_i u_i = 0.$$

A Complex Analytic Structure on M_r

- We now have the tools to define a complex analytic structure on M_r.
- Define the following subspace of $(\mathbb{S}^2)^n$:

$$\tilde{\mathcal{M}}_r = \left\{ \vec{u} \in (\mathbb{S}^2)^n : \sum_{i=1}^{n} r_i u_i = 0 \right\}.$$

- Define $\mathcal{M}_r = \tilde{\mathcal{M}}_r / \mathrm{SO}(3)$.
- If $P \in M_r$, normalizing the edges gives a collection $u_i = e_i / r_i \in \mathbb{S}^2$.
- The closing condition for P to be a polygon requires

$$\sum_{i=1}^{n} r_i u_i = 0.$$

- This gives a homeomorphism

$$\epsilon : M_r \to \mathcal{M}_r$$

called the Gauss map.

A Complex Analytic Structure on M_r

- We will define a complex analytic structure on M_r by recognizing $\mathcal{M}_r$ as a symplectic quotient.

A Complex Analytic Structure on M_r

- We will define a complex analytic structure on M_r by recognizing $\mathcal{M}_r$ as a symplectic quotient.
- Let ω be the standard volume form on $\mathbb{S}^2$, normalized so that

$$\int_{\mathbb{S}^2} \omega = 4\pi.$$

A Complex Analytic Structure on M_r

- We will define a complex analytic structure on M_r by recognizing $\mathcal{M}_r$ as a symplectic quotient.
- Let ω be the standard volume form on $\mathbb{S}^2$, normalized so that

$$\int_{\mathbb{S}^2} \omega = 4\pi.$$

- This is a Kähler form with respect to the standard complex analytic structure and Riemannian metric.

A Complex Analytic Structure on M_r

- We will define a complex analytic structure on M_r by recognizing $\mathcal{M}_r$ as a symplectic quotient.
- Let ω be the standard volume form on $\mathbb{S}^2$, normalized so that

$$\int_{\mathbb{S}^2} \omega = 4\pi.$$

- This is a Kähler form with respect to the standard complex analytic structure and Riemannian metric.
- For each $i = 1, \ldots, n$ let $p_i : (\mathbb{S}^2)^n \to \mathbb{S}^2$ be the projection onto the i^{th} component. We then obtain a symplectic form (that is a Kähler form) Ω on $(\mathbb{S}^2)^n$ by

$$\Omega = \sum_{i=1}^{n} r_i p_i^*(\omega_i).$$

The space $(\mathbb{S}^2)^n$ with the symplectic form Ω is sometimes called the weighted configuration space.

A Complex Analytic Structure on M_r

- The diagonal action of the group $\mathrm{SO}(3)$ on $((\mathbb{S}^2)^n, \Omega)$ is symplectic and Hamiltonian, giving rise to a moment map

$$\mu : (\mathbb{S}^2)^n \to \mathrm{so}(3)^*.$$

A Complex Analytic Structure on M_r

- The diagonal action of the group $\mathrm{SO}(3)$ on $((\mathbb{S}^2)^n, \Omega)$ is symplectic and Hamiltonian, giving rise to a moment map

$$\mu : (\mathbb{S}^2)^n \to \mathrm{so}(3)^*.$$

- Since we have a Lie algebra isomorphism $\mathrm{so}(3) \cong \mathbb{R}^3$, where the Lie bracket on $\mathbb{R}^3$ is the cross product, we can identify $\mathrm{so}(3)^*$ with $(\mathbb{R}^3)^* \cong \mathbb{R}^3$.

A Complex Analytic Structure on M_r

- The diagonal action of the group $\mathrm{SO}(3)$ on $((\mathbb{S}^2)^n, \Omega)$ is symplectic and Hamiltonian, giving rise to a moment map

$$\mu : (\mathbb{S}^2)^n \to \mathrm{so}(3)^*.$$

- Since we have a Lie algebra isomorphism $\mathrm{so}(3) \cong \mathbb{R}^3$, where the Lie bracket on $\mathbb{R}^3$ is the cross product, we can identify $\mathrm{so}(3)^*$ with $(\mathbb{R}^3)^* \cong \mathbb{R}^3$.
- With this identification, the moment map is given by

$$\mu(\vec{u}) = r_1 u_1 + \cdots + r_n u_n,$$

thus $\tilde{M}_r = \mu^{-1}(0)$.

A Complex Analytic Structure on M_r

- The diagonal action of the group $\mathrm{SO}(3)$ on $((\mathbb{S}^2)^n, \Omega)$ is symplectic and Hamiltonian, giving rise to a moment map
$$\mu : (\mathbb{S}^2)^n \to \mathrm{so}(3)^*.$$
- Since we have a Lie algebra isomorphism $\mathrm{so}(3) \cong \mathbb{R}^3$, where the Lie bracket on $\mathbb{R}^3$ is the cross product, we can identify $\mathrm{so}(3)^*$ with $(\mathbb{R}^3)^* \cong \mathbb{R}^3$.
- With this identification, the moment map is given by
$$\mu(\vec{u}) = r_1 u_1 + \cdots + r_n u_n,$$
thus $\tilde{M}_r = \mu^{-1}(0)$.
- The condition $\mu(r) = 0$ is precisely what is required for the polygon to be closed.

A Complex Analytic Structure on M_r

- The diagonal action of the group $\mathrm{SO}(3)$ on $((\mathbb{S}^2)^n, \Omega)$ is symplectic and Hamiltonian, giving rise to a moment map

$$\mu : (\mathbb{S}^2)^n \to \mathrm{so}(3)^*.$$

- Since we have a Lie algebra isomorphism $\mathrm{so}(3) \cong \mathbb{R}^3$, where the Lie bracket on $\mathbb{R}^3$ is the cross product, we can identify $\mathrm{so}(3)^*$ with $(\mathbb{R}^3)^* \cong \mathbb{R}^3$.
- With this identification, the moment map is given by

$$\mu(\vec{u}) = r_1 u_1 + \cdots + r_n u_n,$$

  thus $\tilde{M}_r = \mu^{-1}(0)$.
- The condition $\mu(r) = 0$ is precisely what is required for the polygon to be closed.
- Since $\mathrm{SO}(3)$ is a compact group, it follows that $\mathcal{M}_r = \mu^{-1}(0)/\mathrm{SO}(3)$ is the symplectic reduction of $(\mathbb{S}^2)^n$ via the action of $\mathrm{SO}(3)$.

A Complex Analytic Structure on M_r

- Let Σ be the singularities of $\mathcal{M}_r$.

A Complex Analytic Structure on M_r

- Let Σ be the singularities of $\mathcal{M}_r$.
- $\mathcal{M}_r \setminus \Sigma$ is a smooth symplectic reduction of a complex analytic manifold.

A Complex Analytic Structure on M_r

- Let Σ be the singularities of $\mathcal{M}_r$.
- $\mathcal{M}_r \setminus \Sigma$ is a smooth symplectic reduction of a complex analytic manifold.
- The complex structure on $(\mathbb{S}^2)^n$ is compatible with the symplectic form Ω.

A Complex Analytic Structure on M_r

- Let Σ be the singularities of $\mathcal{M}_r$.
- $\mathcal{M}_r \setminus \Sigma$ is a smooth symplectic reduction of a complex analytic manifold.
- The complex structure on $(\mathbb{S}^2)^n$ is compatible with the symplectic form Ω.
- As a consequence, the symplectic reduction $\mathcal{M}_r \setminus \Sigma$ is complex analytic.

A Complex Analytic Structure on M_r

- Under ϵ^{-1}, the singularities of $\mathcal{M}_r$ correspond to degenerate polygons (line-gon) in M_r.

A Complex Analytic Structure on M_r

- Under ϵ^{-1}, the singularities of $\mathcal{M}_r$ correspond to degenerate polygons (line-gon) in M_r.
- It follows that away from line-gons, M_r has a complex analytic structure.

A Complex Analytic Structure on M_r

- Under ϵ^{-1}, the singularities of $\mathcal{M}_r$ correspond to degenerate polygons (line-gon) in M_r.
- It follows that away from line-gons, M_r has a complex analytic structure.
- The singularities are isolated, there are finitely many, and are contained in compatible complex analytic neighborhoods.

A Complex Analytic Structure on M_r

- Under ϵ^{-1}, the singularities of $\mathcal{M}_r$ correspond to degenerate polygons (line-gon) in M_r.
- It follows that away from line-gons, M_r has a complex analytic structure.
- The singularities are isolated, there are finitely many, and are contained in compatible complex analytic neighborhoods.
- This makes the moduli space M_r into a complex analytic space that is obtained as a result of symplectic reduction.

Geometric Invariant Theory

- The algebraic side of this story comes in the form of Mumford's geometric invariant theory.

Geometric Invariant Theory

- The algebraic side of this story comes in the form of Mumford's geometric invariant theory.
- The setting is an algebraic group (such as a matrix Lie group) G acting on an algebraic variety X.

Geometric Invariant Theory

- The algebraic side of this story comes in the form of Mumford's geometric invariant theory.
- The setting is an algebraic group (such as a matrix Lie group) G acting on an algebraic variety X.
- We would like to find a way of taking a quotient of X by the action of G and having the orbit space be an algebraic variety.

Geometric Invariant Theory

- The algebraic side of this story comes in the form of Mumford's geometric invariant theory.
- The setting is an algebraic group (such as a matrix Lie group) G acting on an algebraic variety X.
- We would like to find a way of taking a quotient of X by the action of G and having the orbit space be an algebraic variety.
- The problem is that the ordinary quotient may not be an algebraic variety - it may not even be Hausdorff.

GIT, An Example

- Suppose $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$ where the action is

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2).$$

GIT, An Example

- Suppose $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$ where the action is

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2).$$

- If $z_1 \neq 0$ and $z_2 \neq 0$, the orbits have the form $z_1 z_2 = c$ where $c \in \mathbb{C}^*$.
- For $z_1 z_2 = 0$, there are three orbits - the orbit of $(0, z_2)$ where $z_2 \neq 0$, the orbit of $(z_1, 0)$ where $z_1 \neq 0$ and the orbit of $(0, 0)$.
- The latter consists of the single point $\{(0, 0)\}$ whereas the other two orbits are the z_2- and z_1-axes minus the origin, respectively.

GIT, An Example

- Suppose $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$ where the action is

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2).$$

- If $z_1 \neq 0$ and $z_2 \neq 0$, the orbits have the form $z_1 z_2 = c$ where $c \in \mathbb{C}^*$.
- For $z_1 z_2 = 0$, there are three orbits - the orbit of $(0, z_2)$ where $z_2 \neq 0$, the orbit of $(z_1, 0)$ where $z_1 \neq 0$ and the orbit of $(0, 0)$.
- The latter consists of the single point $\{(0,0)\}$ whereas the other two orbits are the z_2- and z_1-axes minus the origin, respectively.
- Any open neighborhood of the orbit of $(0, 0)$ in the orbit space must intersect other orbits, and the orbit space is therefore not Hausdorff.

Geometric Invariant Theory

- Geometric invariant theory constructs another form of quotient to get around such problems.

Geometric Invariant Theory

- Geometric invariant theory constructs another form of quotient to get around such problems.
- As a matter of notation, the GIT quotient is denoted by $X//G$.

Geometric Invariant Theory

- Geometric invariant theory constructs another form of quotient to get around such problems.
- As a matter of notation, the GIT quotient is denoted by $X//G$.
- In the example above, the GIT quotient is simply $\mathbb{C}$.

Geometric Invariant Theory

- Geometric invariant theory constructs another form of quotient to get around such problems.
- As a matter of notation, the GIT quotient is denoted by $X//G$.
- In the example above, the GIT quotient is simply $\mathbb{C}$.
- The basic construction involves identifying the ring of G-invariants of X.

Geometric Invariant Theory

- Geometric invariant theory constructs another form of quotient to get around such problems.
- As a matter of notation, the GIT quotient is denoted by $X//G$.
- In the example above, the GIT quotient is simply $\mathbb{C}$.
- The basic construction involves identifying the ring of G-invariants of X.
- We begin with the affine case.

The Affine Case

- Suppose X is an affine variety defined by polynomials $p_1, \ldots, p_k$ in n complex variables.

The Affine Case

- Suppose X is an affine variety defined by polynomials $p_1, \ldots, p_k$ in n complex variables.
- Let $\mathcal{R}$ be the ring of functions on X:

$$\mathcal{R} = \mathbb{C}[z_1, \ldots, z_n]/(p_1, \ldots, p_k).$$

This gives us a natural embedding $X \hookrightarrow \mathbb{C}^n$ as the space of all points $z \in \mathbb{C}^n$ where $p_i(z) = 0$ for all $i = 1, \ldots, k$.

The Affine Case

- Suppose X is an affine variety defined by polynomials $p_1, \ldots, p_k$ in n complex variables.
- Let $\mathcal{R}$ be the ring of functions on X:

$$\mathcal{R} = \mathbb{C}[z_1, \ldots, z_n]/(p_1, \ldots, p_k).$$

  This gives us a natural embedding $X \hookrightarrow \mathbb{C}^n$ as the space of all points $z \in \mathbb{C}^n$ where $p_i(z) = 0$ for all $i = 1, \ldots, k$.
- In this setting, X is called the spectrum of $\mathcal{R}$ and we write

$$X = \mathrm{Spec}(\mathcal{R}).$$

  (This is equivalent to the set of all prime ideals of $\mathcal{R}$.)

The Affine Case

- Suppose X is an affine variety defined by polynomials $p_1, \ldots, p_k$ in n complex variables.
- Let $\mathcal{R}$ be the ring of functions on X:

$$\mathcal{R} = \mathbb{C}[z_1, \ldots, z_n]/(p_1, \ldots, p_k).$$

  This gives us a natural embedding $X \hookrightarrow \mathbb{C}^n$ as the space of all points $z \in \mathbb{C}^n$ where $p_i(z) = 0$ for all $i = 1, \ldots, k$.
- In this setting, X is called the spectrum of $\mathcal{R}$ and we write

$$X = \mathrm{Spec}(\mathcal{R}).$$

  (This is equivalent to the set of all prime ideals of $\mathcal{R}$.)
- If $f : \mathcal{R} \to \mathcal{S}$ is a ring homomorphism and $\mathfrak{p} \subset \mathcal{S}$ is a prime ideal, $f^{-1}(\mathfrak{p})$ is a prime ideal of $\mathcal{R}$, which induces a map from $\mathrm{Spec}(\mathcal{S})$ to $\mathrm{Spec}(\mathcal{R})$.

- Now suppose G is a complex Lie group that acts on our ring $\mathcal{R}$ by ring automorphisms.

The Affine Case

- Now suppose G is a complex Lie group that acts on our ring $\mathcal{R}$ by ring automorphisms.
- The action preserves prime ideals and hence acts on $\mathrm{Spec}(\mathcal{R})$, so G acts on X.

The Affine Case

- Now suppose G is a complex Lie group that acts on our ring $\mathcal{R}$ by ring automorphisms.
- The action preserves prime ideals and hence acts on $\mathrm{Spec}(\mathcal{R})$, so G acts on X.
- A function $f \in \mathcal{R}$ on X is G-invariant if $g \cdot f(x) = f(x)$ for all $g \in G$ and $x \in X$.

The Affine Case

- Now suppose G is a complex Lie group that acts on our ring $\mathcal{R}$ by ring automorphisms.
- The action preserves prime ideals and hence acts on $\mathrm{Spec}(\mathcal{R})$, so G acts on X.
- A function $f \in \mathcal{R}$ on X is G-invariant if $g \cdot f(x) = f(x)$ for all $g \in G$ and $x \in X$.
- Denote the set of all G-invariant functions on X by $\mathcal{R}^G$.

The Affine Case

- Now suppose G is a complex Lie group that acts on our ring $\mathcal{R}$ by ring automorphisms.
- The action preserves prime ideals and hence acts on $\mathrm{Spec}(\mathcal{R})$, so G acts on X.
- A function $f \in \mathcal{R}$ on X is G-invariant if $g \cdot f(x) = f(x)$ for all $g \in G$ and $x \in X$.
- Denote the set of all G-invariant functions on X by $\mathcal{R}^G$.
- $\mathcal{R}^G$ forms a ring (with operations inherited from $\mathcal{R}$) called the ring of invariants. We define $\mathrm{Spec}(\mathcal{R}^G)$ to be the GIT quotient of X by G:

$$X//G = \mathrm{Spec}(\mathcal{R}^G).$$

The Affine Case

- Let us return to the example above where $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$.
- Since X is an affine variety defined solely by the zero polynomial, the ring of functions is $\mathcal{R} = \mathbb{C}[z_1, z_2]$, and the ring of invariants is $\mathcal{R}^G$, generated by the polynomial $z_1 z_2$.

The Affine Case

- Let us return to the example above where $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$.
- Since X is an affine variety defined solely by the zero polynomial, the ring of functions is $\mathcal{R} = \mathbb{C}[z_1, z_2]$, and the ring of invariants is $\mathcal{R}^G$, generated by the polynomial $z_1 z_2$.
- If we let $w = z_1 z_2$:

$$\mathcal{R}^G = \mathbb{C}[w].$$

The Affine Case

- Let us return to the example above where $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$.
- Since X is an affine variety defined solely by the zero polynomial, the ring of functions is $\mathcal{R} = \mathbb{C}[z_1, z_2]$, and the ring of invariants is $\mathcal{R}^G$, generated by the polynomial $z_1 z_2$.
- If we let $w = z_1 z_2$:

$$\mathcal{R}^G = \mathbb{C}[w].$$

- It follows that $X//G = \mathbb{C}$.

The Affine Case

- Let us return to the example above where $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$.
- Since X is an affine variety defined solely by the zero polynomial, the ring of functions is $\mathcal{R} = \mathbb{C}[z_1, z_2]$, and the ring of invariants is $\mathcal{R}^G$, generated by the polynomial $z_1 z_2$.
- If we let $w = z_1 z_2$:

$$\mathcal{R}^G = \mathbb{C}[w].$$

- It follows that $X//G = \mathbb{C}$.
- What has happened geometrically is the orbits where $z_1 z_2 \neq 0$ have been left unaltered, but we have identified the three orbits where $z_1 z_2 = 0$.

The Projective Case

- To see why this construction isn't sufficient for the projective case, let $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$ as above, but let the action be given by

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2).$$

The Projective Case

- To see why this construction isn't sufficient for the projective case, let $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$ as above, but let the action be given by

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2).$$

- The ring of functions $\mathcal{R} = \mathbb{C}[z_1, z_2]$, but the only invariant polynomials are constants and therefore $\mathrm{Spec}(\mathcal{R}^G) = X//G$ is just a single point.

The Projective Case

- To see why this construction isn't sufficient for the projective case, let $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$ as above, but let the action be given by

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2).$$

- The ring of functions $\mathcal{R} = \mathbb{C}[z_1, z_2]$, but the only invariant polynomials are constants and therefore $\mathrm{Spec}(\mathcal{R}^G) = X//G$ is just a single point.
- The obvious quotient *should* be $\mathbb{CP}^1$, but projective space is not $\mathrm{Spec}(\mathcal{S})$ for any ring $\mathcal{S}$.

The Projective Case

- To see why this construction isn't sufficient for the projective case, let $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$ as above, but let the action be given by

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2).$$

- The ring of functions $\mathcal{R} = \mathbb{C}[z_1, z_2]$, but the only invariant polynomials are constants and therefore $\mathrm{Spec}(\mathcal{R}^G) = X//G$ is just a single point.
- The obvious quotient *should* be $\mathbb{CP}^1$, but projective space is not $\mathrm{Spec}(\mathcal{S})$ for any ring $\mathcal{S}$.
- We need to do more work to obtain a GIT quotient for a projective variety.

The Projective Case

- Suppose X is a projective variety. The first construction involves an embedding of X into $\mathbb{CP}^n$ for some sufficiently large n.

The Projective Case

- Suppose X is a projective variety. The first construction involves an embedding of X into $\mathbb{CP}^n$ for some sufficiently large n.
- In the affine case, the corresponding embedding was handled using the ring of functions on X.

The Projective Case

- Suppose X is a projective variety. The first construction involves an embedding of X into $\mathbb{CP}^n$ for some sufficiently large n.
- In the affine case, the corresponding embedding was handled using the ring of functions on X.
- We don't have that at our disposal in the projective case, since such functions would have to be constant.

The Projective Case

- Let $\mathcal{L}$ be a line bundle over X.
- Let $\Gamma(X, \mathcal{L})$ be the space of sections of $\mathcal{L}$.
- Let $(s_0, \ldots, s_n)$ be a set of its sections.

The Projective Case

- Let $\mathcal{L}$ be a line bundle over X.
- Let $\Gamma(X, \mathcal{L})$ be the space of sections of $\mathcal{L}$.
- Let $(s_0, \ldots, s_n)$ be a set of its sections.
- We say that $\mathcal{L}$ is *very ample* if for every $x \in X$ there is i such that $s_i(x) \neq 0$.

The Projective Case

- Let $\mathcal{L}$ be a line bundle over X.
- Let $\Gamma(X, \mathcal{L})$ be the space of sections of $\mathcal{L}$.
- Let $(s_0, \ldots, s_n)$ be a set of its sections.
- We say that $\mathcal{L}$ is *very ample* if for every $x \in X$ there is i such that $s_i(x) \neq 0$.
- This allows us to construct a closed embedding

$$X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{L})^*).$$

The Projective Case

- Let $\mathcal{L}$ be a line bundle over X.
- Let $\Gamma(X, \mathcal{L})$ be the space of sections of $\mathcal{L}$.
- Let $(s_0, \ldots, s_n)$ be a set of its sections.
- We say that $\mathcal{L}$ is *very ample* if for every $x \in X$ there is i such that $s_i(x) \neq 0$.
- This allows us to construct a closed embedding

$$X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{L})^*).$$

- Given $x \in X$ we map

$$x \mapsto [f_x] = [s_0(x) : s_1(x) : \cdots : s_n(x)].$$

- This is well defined since there is some i such that $s_i(x) \neq 0$.

The Projective Case

- Let $\mathcal{L}$ be a line bundle over X.
- Let $\Gamma(X, \mathcal{L})$ be the space of sections of $\mathcal{L}$.
- Let $(s_0, \ldots, s_n)$ be a set of its sections.
- We say that $\mathcal{L}$ is *very ample* if for every $x \in X$ there is i such that $s_i(x) \neq 0$.
- This allows us to construct a closed embedding

$$X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{L})^*).$$

- Given $x \in X$ we map

$$x \mapsto [f_x] = [s_0(x) : s_1(x) : \cdots : s_n(x)].$$

- This is well defined since there is some i such that $s_i(x) \neq 0$.
- Identifying $\mathbb{P}(\Gamma(X, \mathcal{L})^*)$ with $\mathbb{CP}^n$, this gives us a closed embedding $X \hookrightarrow \mathbb{CP}^n$.

The Projective Case

- Suppose we have a Lie group G acting on X.
- Suppose the action lifts to an action of G on $\mathcal{L}$ that is linear on fibers and takes the fiber L_x to the fiber $L_{g \cdot x}$ for every $g \in G$ and $x \in X$.

The Projective Case

- Suppose we have a Lie group G acting on X.
- Suppose the action lifts to an action of G on $\mathcal{L}$ that is linear on fibers and takes the fiber L_x to the fiber $L_{g \cdot x}$ for every $g \in G$ and $x \in X$.
- We say that the action of G on $\mathcal{L}$ linearizes the action of G on X, and that $\mathcal{L}$ is a linearized line bundle.

The Projective Case

- Suppose we have a Lie group G acting on X.
- Suppose the action lifts to an action of G on $\mathcal{L}$ that is linear on fibers and takes the fiber L_x to the fiber $L_{g \cdot x}$ for every $g \in G$ and $x \in X$.
- We say that the action of G on $\mathcal{L}$ linearizes the action of G on X, and that $\mathcal{L}$ is a linearized line bundle.
- The linearization of the action preserves sections. As a consequence, we get an action of G on $\mathbb{P}(\Gamma(X, \mathcal{L})^*)$ such that the embedding $X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{L})^*)$ is G-equivariant.

The Projective Case

- We can thus consider G acting on X sitting inside of $\mathbb{CP}^n$.

The Projective Case

- We can thus consider G acting on X sitting inside of $\mathbb{CP}^n$.
- If we let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the natural projection, then $\pi^{-1}(X)$ is an affine variety in $\mathbb{C}^{n+1}$ that is invariant under dilations.

The Projective Case

- We can thus consider G acting on X sitting inside of $\mathbb{CP}^n$.
- If we let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the natural projection, then $\pi^{-1}(X)$ is an affine variety in $\mathbb{C}^{n+1}$ that is invariant under dilations.
- The action of G lifts to an action on $\pi^{-1}(X)$, so we can consider the ring of **homogeneous** invariants.

The Projective Case

- We can thus consider G acting on X sitting inside of $\mathbb{CP}^n$.
- If we let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the natural projection, then $\pi^{-1}(X)$ is an affine variety in $\mathbb{C}^{n+1}$ that is invariant under dilations.
- The action of G lifts to an action on $\pi^{-1}(X)$, so we can consider the ring of **homogeneous** invariants.
- Taking the quotient again gives us a projective variety, and provides us with our construction of $X//G$.

The Projective Case

- In order to be able to prove anything about projective GIT quotients, we need to make this construction more concrete.
- A point $x \in X$ is called *semi-stable* (with respect to $\mathcal{L}$) if there is some $m > 0$ and G-invariant section $s \in \Gamma(X, \mathcal{L}^{\otimes m})^G$ such that $s(x) \neq 0$ and the set $X_s = \{y \in X : s(y) \neq 0\}$ is affine.
- If additionally the orbit of G through x is closed, x is said to be *stable* (with respect to $\mathcal{L}$).
- Note that if x is stable with $s(x) \neq 0$, the orbit of G through any point $y \in X_s$ is closed.

The Projective Case

- $x \in X$ that is not semi-stable is *unstable* (with respect to $\mathcal{L}$).

The Projective Case

- $x \in X$ that is not semi-stable is *unstable* (with respect to $\mathcal{L}$).
- The set of semistable, stable, and unstable points are denoted by

$$X^{\mathrm{ss}}(\mathcal{L}),\ X^{\mathrm{s}}(\mathcal{L}),\ X^{\mathrm{us}}(\mathcal{L})$$

respectively.

The Projective Case

- $x \in X$ that is not semi-stable is *unstable* (with respect to $\mathcal{L}$).
- The set of semistable, stable, and unstable points are denoted by

$$X^{\mathrm{ss}}(\mathcal{L}),\ X^{\mathrm{s}}(\mathcal{L}),\ X^{\mathrm{us}}(\mathcal{L})$$

  respectively.
- It is clear that $X^{\mathrm{ss}}(\mathcal{L})$ and $X^{\mathrm{s}}(\mathcal{L})$ are open subsets of X, although they may be empty.

The Projective Case

- To construct the GIT quotient, we begin with $X \setminus X^{\mathrm{us}} = X^{\mathrm{ss}}$.

The Projective Case

- To construct the GIT quotient, we begin with $X \setminus X^{\mathrm{us}} = X^{\mathrm{ss}}$.
- We can cover X^{ss} with the affine open subsets X_{s_i}, take the affine GIT quotient of these open subsets and "glue them together" in an appropriate manner. This gives us the GIT quotient

$$X^{\mathrm{ss}}//G$$

which is what we call $X//G$.

- To construct the GIT quotient, we begin with $X \setminus X^{\mathrm{us}} = X^{\mathrm{ss}}$.
- We can cover X^{ss} with the affine open subsets X_{s_i}, take the affine GIT quotient of these open subsets and "glue them together" in an appropriate manner. This gives us the GIT quotient

$$X^{\mathrm{ss}}//G$$

  which is what we call $X//G$.
- This construction makes $X//S$ into a projective variety. What actually happens is that the orbits of stable points in X become single points in $X//G$, while orbits of points in $X^{ss} \setminus X^{s}$ are identified whenever the intersection of their closures is nonempty.

Kirwen-Kempf-Ness Theorem

- The Kirwan-Kempf-Ness Theorem connects GIT quotients to symplectic reduction on symplectic projective varieties.

Kirwen-Kempf-Ness Theorem

- The Kirwan-Kempf-Ness Theorem connects GIT quotients to symplectic reduction on symplectic projective varieties.
- We suppose X is a projective variety with symplectic form ω and a line bundle $\mathcal{L}$ such that $\mathrm{c}_1(\mathcal{L}) = [\omega]$ in $H^2_{\mathrm{dR}}(X)$, where c_1 is the first Chern class. Suppose further that G is a Lie group acting on X that linearizes $\mathcal{L}$.

Kirwen-Kempf-Ness Theorem

- The Kirwan-Kempf-Ness Theorem connects GIT quotients to symplectic reduction on symplectic projective varieties.
- We suppose X is a projective variety with symplectic form ω and a line bundle $\mathcal{L}$ such that $\mathrm{c}_1(\mathcal{L}) = [\omega]$ in $H^2_{\mathrm{dR}}(X)$, where c_1 is the first Chern class. Suppose further that G is a Lie group acting on X that linearizes $\mathcal{L}$.
- Let $\mathfrak{g}$ be the Lie algebra over G and suppose the action of G admits a moment map $\mu : X \to \mathfrak{g}^*$.

Kirwen-Kempf-Ness Theorem

- The first part of the Kirwan-Kempf-Ness theorem says that $\mu^{-1}(0) \subset X^{\mathrm{ss}}$.

- The first part of the Kirwan-Kempf-Ness theorem says that $\mu^{-1}(0) \subset X^{\mathrm{ss}}$.
- Now let $K \subset G$ be a maximal compact subgroup of G.

Kirwen-Kempf-Ness Theorem

- The first part of the Kirwan-Kempf-Ness theorem says that $\mu^{-1}(0) \subset X^{\mathrm{ss}}$.
- Now let $K \subset G$ be a maximal compact subgroup of G.
- The second part of the Kirwan-Kempf-Ness theorem says that the inclusion $\mu^{-1}(0) \hookrightarrow X^{\mathrm{ss}}$ induces a homeomorphism

$$\mu^{-1}(0)/K \cong X^{ss}//G.$$

Kirwen-Kempf-Ness Theorem

- The first part of the Kirwan-Kempf-Ness theorem says that $\mu^{-1}(0) \subset X^{\mathrm{ss}}$.
- Now let $K \subset G$ be a maximal compact subgroup of G.
- The second part of the Kirwan-Kempf-Ness theorem says that the inclusion $\mu^{-1}(0) \hookrightarrow X^{\mathrm{ss}}$ induces a homeomorphism

$$\mu^{-1}(0)/K \cong X^{ss}//G.$$

- If $\mu^{-1}(0)/K$ is smooth, this is a diffeomorphism. If $\mu^{-1}(0)/K$ is complex analytic, this is a complex analytic equivalence.

- We would like to find a complex-analytic equivalence between the moduli space M_r and the weighted configuration space $((\mathbb{S}^2)^n, \Omega)$.

- We would like to find a complex-analytic equivalence between the moduli space M_r and the weighted configuration space $((\mathbb{S}^2)^n, \Omega)$.
- We will do this using the Kirwan-Kempf-Ness theorem.
- Denote $((\mathbb{S}^2)^n, \Omega)$ by M.

Polygon Spaces

- We would like to find a complex-analytic equivalence between the moduli space M_r and the weighted configuration space $((\mathbb{S}^2)^n, \Omega)$.
- We will do this using the Kirwan-Kempf-Ness theorem.
- Denote $((\mathbb{S}^2)^n, \Omega)$ by M.
- The underlying space is a product of copies of $\mathbb{CP}^1$, M is a projective variety

Polygon Spaces

- We would like to find a complex-analytic equivalence between the moduli space M_r and the weighted configuration space $((\mathbb{S}^2)^n, \Omega)$.
- We will do this using the Kirwan-Kempf-Ness theorem.
- Denote $((\mathbb{S}^2)^n, \Omega)$ by M.
- The underlying space is a product of copies of $\mathbb{CP}^1$, M is a projective variety
- We will form a GIT quotient structure on M. The construction will depend on our choice of r.

Polygon Spaces

- We would like to find a complex-analytic equivalence between the moduli space M_r and the weighted configuration space $((\mathbb{S}^2)^n, \Omega)$.
- We will do this using the Kirwan-Kempf-Ness theorem.
- Denote $((\mathbb{S}^2)^n, \Omega)$ by M.
- The underlying space is a product of copies of $\mathbb{CP}^1$, M is a projective variety
- We will form a GIT quotient structure on M. The construction will depend on our choice of r.
- We need to assume $r \in \mathbb{Q}^n_+$

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- $G = \mathrm{PSL}(2, \mathbb{C})$ acts on $\mathbb{S}^2$, and thus extends to a diagonal action on M by
$$A \cdot (u_1, \ldots, u_n) = (A \cdot u_1, \ldots, A \cdot u_n).$$

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- $G = \mathrm{PSL}(2, \mathbb{C})$ acts on $\mathbb{S}^2$, and thus extends to a diagonal action on M by
$$A \cdot (u_1, \ldots, u_n) = (A \cdot u_1, \ldots, A \cdot u_n).$$

- Given the action and $r \in \mathbb{D}_2^n$, the semi-stable points in M are the points $\vec{u} \in (\mathbb{S}^2)^n$ such that
$$\sum_{u_j = v} r_j \leq 1$$
for every $v \in \mathbb{S}^2$.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- stable points in M are those that satisfy the same condition with a strict inequality.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- stable points in M are those that satisfy the same condition with a strict inequality.
- Let $Q = M^{\mathrm{ss}}//\mathrm{PSL}(2, \mathbb{C})$.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- stable points in M are those that satisfy the same condition with a strict inequality.
- Let $Q = M^{\text{ss}}//\mathrm{PSL}(2, \mathbb{C})$.
- Q is the GIT quotient $M//G$ and is therefore a projective variety.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- Define the cusp points to be $M^{\mathrm{cusp}} = M^{\mathrm{ss}} \setminus M^{\mathrm{s}}$.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- Define the cusp points to be $M^{\mathrm{cusp}} = M^{\mathrm{ss}} \setminus M^{\mathrm{s}}$.
- Every point in M^{cusp} is determined by a partition of $S = \{1, \ldots, n\}$ into two disjoint sets $S_1 = \{i_1, \ldots, i_j\}$ and $S_2 = \{j_1, \ldots, j_{n-k}\}$ such that $r_{i_1} + \cdots + r_{i_k} = 1$ (and whence $r_{j_1} + \cdots + r_{j_{n-k}} = 1$).

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- Define the cusp points to be $M^{\text{cusp}} = M^{\text{ss}} \setminus M^{\text{s}}$.
- Every point in M^{cusp} is determined by a partition of $S = \{1, \ldots, n\}$ into two disjoint sets $S_1 = \{i_1, \ldots, i_j\}$ and $S_2 = \{j_1, \ldots, j_{n-k}\}$ such that $r_{i_1} + \cdots + r_{i_k} = 1$ (and whence $r_{j_1} + \cdots + r_{j_{n-k}} = 1$).
- In the quotient, the cusp points are *uniquely* determined by the corresponding partition. Therefore, the subspace $Q^{\text{cusp}} = M^{\text{cusp}}/G \subset Q$ is in one-to-one correspondence with a subset of partitions of n, which in turn is finite.
- Thus, in the quotient, there are at most finitely many cusp points.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- The closing condition

$$\sum_{i=1}^{n} r_i u_i = 0$$

implies that $\tilde{\mathcal{M}}_r \subset M^{\mathrm{ss}}$.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- The closing condition
$$\sum_{i=1}^{n} r_i u_i = 0$$
implies that $\tilde{\mathcal{M}}_r \subset M^{\mathrm{ss}}$.
- This is the first part of Kirwan-Kempf-Ness: $\mu^{-1}(0) \subset M^{\mathrm{ss}}$.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- $\mathrm{SO}(3) \subset \mathrm{PSL}(2, \mathbb{C})$ is a maximal compact subgroup so there is a complex analytic equivalence

$$\mathcal{M}_r \cong Q$$

from the second part of the Kirwan-Kempf-Ness, which induces, via the Gauss map ϵ, a complex analytic equivalence

$$M_r \cong Q.$$

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- $\mathrm{SO}(3) \subset \mathrm{PSL}(2,\mathbb{C})$ is a maximal compact subgroup so there is a complex analytic equivalence

$$\mathcal{M}_r \cong Q$$

from the second part of the Kirwan-Kempf-Ness, which induces, via the Gauss map ϵ, a complex analytic equivalence

$$M_r \cong Q.$$

- Under ϵ, the cusp points Q^{cusp} correspond to the degenerate polygons in M_r.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- $\mathrm{SO}(3) \subset \mathrm{PSL}(2, \mathbb{C})$ is a maximal compact subgroup so there is a complex analytic equivalence

$$\mathcal{M}_r \cong Q$$

from the second part of the Kirwan-Kempf-Ness, which induces, via the Gauss map ϵ, a complex analytic equivalence

$$M_r \cong Q.$$

- Under ϵ, the cusp points Q^{cusp} correspond to the degenerate polygons in M_r.
- It follows that M_r, and hence $\mathcal{M}_r$, have at most finitely many singularities.

Moduli Spaces for Favorable Chambers

- Consider the moduli spaces for special vectors $r \in \mathbb{D}_2^n$.

Moduli Spaces for Favorable Chambers

- Consider the moduli spaces for special vectors $r \in \mathbb{D}_2^n$.
- The hypersimplex is divided into chambers by walls of the form

$$W_J = \left\{ (x_1, \ldots, x_n) \in \mathbb{D}_2^n : \sum_{i \in J} x_i = 1 \right\}$$

where J runs over all proper subsets of $\{1, \ldots, n\}$.

Moduli Spaces for Favorable Chambers

- The chambers are polytopes

Moduli Spaces for Favorable Chambers

- The chambers are polytopes
- Two points x and y are in the same chamber if and only if for every proper subset $J \subset \{1, \ldots, n\}$

$$\sum_J x_j \leq 1 \iff \sum_J y_j \leq 1$$

Moduli Spaces for Favorable Chambers

- The chambers are polytopes
- Two points x and y are in the same chamber if and only if for every proper subset $J \subset \{1, \ldots, n\}$

$$\sum_J x_j \leq 1 \iff \sum_J y_j \leq 1$$

- If the chamber is maximal (with respect to inclusion), the inequalities are strict

Moduli Spaces for Favorable Chambers

- For each $i = 1, \ldots, n$, consider the wall $W_{\{i\}}$.

Moduli Spaces for Favorable Chambers

- For each $i = 1, \ldots, n$, consider the wall $W_{\{i\}}$.
- Let Δ_i be the unique maximal chamber in $\mathbb{D}_2^n$ containing $W_{\{i\}}$.

Moduli Spaces for Favorable Chambers

- For each $i = 1, \ldots, n$, consider the wall $W_{\{i\}}$.
- Let Δ_i be the unique maximal chamber in $\mathbb{D}_2^n$ containing $W_{\{i\}}$.
- A theorem of Hu states that if r is in the interior of Δ_i, the moduli space M_r is isomorphic to $\mathbb{CP}^{n-3}$.
- The quadrilateral example in the introduction is a moduli space isomorphic to $\mathbb{CP}^{n-3}$. Since $n = 4$ for a quadrilateral, this is the same as $\mathbb{CP}^1$, which is isomorphic to $\mathbb{S}^2$.

Moduli Spaces for Favorable Chambers

- For each $i = 1, \ldots, n$, consider the wall $W_{\{i\}}$.
- Let Δ_i be the unique maximal chamber in $\mathbb{D}_2^n$ containing $W_{\{i\}}$.
- A theorem of Hu states that if r is in the interior of Δ_i, the moduli space M_r is isomorphic to $\mathbb{CP}^{n-3}$.
- The quadrilateral example in the introduction is a moduli space isomorphic to $\mathbb{CP}^{n-3}$. Since $n = 4$ for a quadrilateral, this is the same as $\mathbb{CP}^1$, which is isomorphic to $\mathbb{S}^2$.
- The discussion in the introduction shows that any $r \in \mathbb{R}_+^n$ where there is some i such that r_i is sufficiently larger then the other r_j's is contained in $\Delta_{\{i\}}$, hence $M_r \cong \mathbb{CP}^{n-3}$.

Stable Polygons and Stable Curves

- The moduli spaces M_r may have finitely many singularities, which can be resolved. There is a mechanism for resolving these singularities that applies to M_r for every r, which is the second result of Hu we use.

Stable Polygons and Stable Curves

- The moduli spaces M_r may have finitely many singularities, which can be resolved. There is a mechanism for resolving these singularities that applies to M_r for every r, which is the second result of Hu we use.
- The mechanism grows out of the notion of stable polygons, which is connected to the moduli space of stable n-pointed curves of genus zero.

Stable Curves of Genus Zero

- Start with the moduli space of n distinct points on $(\mathbb{CP}^1)^n$ modulo the action of $\mathrm{PSL}(2, \mathbb{C})$, denoted by $\mathcal{M}_{0,n}$. This will contain all the generic points in the moduli space we will construct.

Stable Curves of Genus Zero

- Start with the moduli space of n distinct points on $(\mathbb{CP}^1)^n$ modulo the action of $\mathrm{PSL}(2, \mathbb{C})$, denoted by $\mathcal{M}_{0,n}$. This will contain all the generic points in the moduli space we will construct.
- $\mathcal{M}_{0,n}$ is not compact, so we will add in enough points to form a compactification.

Stable Curves of Genus Zero

- Start with the moduli space of n distinct points on $(\mathbb{CP}^1)^n$ modulo the action of $\mathrm{PSL}(2, \mathbb{C})$, denoted by $\mathcal{M}_{0,n}$. This will contain all the generic points in the moduli space we will construct.
- $\mathcal{M}_{0,n}$ is not compact, so we will add in enough points to form a compactification.
- View a point in $\mathcal{M}_{0,n}$ as n distinct points on a single copy of $\mathbb{CP}^1$ and we let these points vary. As a pair of points get close to one another, introduce a new copy of $\mathbb{CP}^1$ intersecting the original $\mathbb{CP}^1$ at the point of intersection.

Stable Curves of Genus Zero

- Start with the moduli space of n distinct points on $(\mathbb{CP}^1)^n$ modulo the action of $\mathrm{PSL}(2,\mathbb{C})$, denoted by $\mathcal{M}_{0,n}$. This will contain all the generic points in the moduli space we will construct.
- $\mathcal{M}_{0,n}$ is not compact, so we will add in enough points to form a compactification.
- View a point in $\mathcal{M}_{0,n}$ as n distinct points on a single copy of $\mathbb{CP}^1$ and we let these points vary. As a pair of points get close to one another, introduce a new copy of $\mathbb{CP}^1$ intersecting the original $\mathbb{CP}^1$ at the point of intersection.
- The colliding points will then proceed along the new $\mathbb{CP}^1$.

Stable Curves of Genus Zero

- The end result will be a collection of intersecting copies of $\mathbb{CP}^1$ with the extra restriction that each irreducible component (each copy of $\mathbb{CP}^1$) has at least three points consisting of the original points that are left on that copy of $\mathbb{CP}^1$ and intersections with other copies.

Stable Curves of Genus Zero

- The end result will be a collection of intersecting copies of $\mathbb{CP}^1$ with the extra restriction that each irreducible component (each copy of $\mathbb{CP}^1$) has at least three points consisting of the original points that are left on that copy of $\mathbb{CP}^1$ and intersections with other copies.
- The process stops after finitely many steps.

Stable Curves of Genus Zero

- The end result will be a collection of intersecting copies of $\mathbb{CP}^1$ with the extra restriction that each irreducible component (each copy of $\mathbb{CP}^1$) has at least three points consisting of the original points that are left on that copy of $\mathbb{CP}^1$ and intersections with other copies.
- The process stops after finitely many steps.
- The number "three" is to prevent the existence of any nontrivial linear fraction transformation that can act on our space.

Stable Curves of Genus Zero

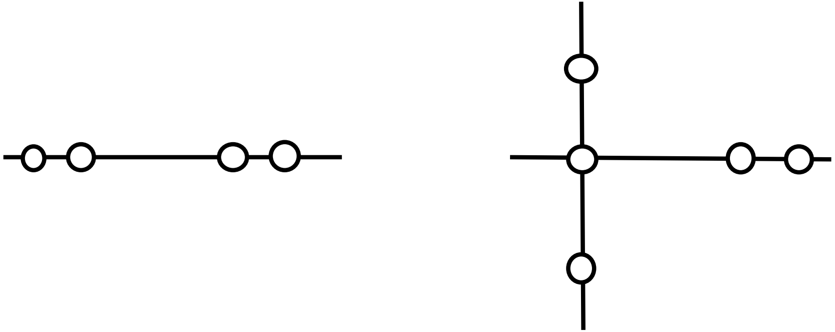
- The end result will be a collection of intersecting copies of $\mathbb{CP}^1$ with the extra restriction that each irreducible component (each copy of $\mathbb{CP}^1$) has at least three points consisting of the original points that are left on that copy of $\mathbb{CP}^1$ and intersections with other copies.
- The process stops after finitely many steps.
- The number "three" is to prevent the existence of any nontrivial linear fraction transformation that can act on our space.
- This creates stable n-pointed curves of genus zero, and the moduli space of such stable curves is denoted $\overline{\mathcal{M}}_{0,n}$. "genus zero" comes from the fact that we constructed the curves out of $\mathbb{CP}^1$.

Stable Curves of Genus Zero

- The end result will be a collection of intersecting copies of $\mathbb{CP}^1$ with the extra restriction that each irreducible component (each copy of $\mathbb{CP}^1$) has at least three points consisting of the original points that are left on that copy of $\mathbb{CP}^1$ and intersections with other copies.
- The process stops after finitely many steps.
- The number "three" is to prevent the existence of any nontrivial linear fraction transformation that can act on our space.
- This creates stable *n*-pointed curves of genus zero, and the moduli space of such stable curves is denoted $\overline{\mathcal{M}}_{0,n}$. "genus zero" comes from the fact that we constructed the curves out of $\mathbb{CP}^1$.
- $\overline{\mathcal{M}}_{0,n}$ is a compactification of $\mathcal{M}_{0,n}$ that illustrates a general procedure called the Deligne-Mumford compactification

Stable Curves of Genus Zero, an Example

- As an example, consider four points on $\mathbb{CP}^1$. A stable curve is illustrated in the cartoon below, where each line represents a copy of $\mathbb{CP}^1$ and each circle either represents a point or an intersection.



Stable Polygons

- We do a similar construction to build stable polygons, but our building blocks will be a little bit more sensitive so a certain amount of care must be taken.

Stable Polygons

- We do a similar construction to build stable polygons, but our building blocks will be a little bit more sensitive so a certain amount of care must be taken.
- An n-gon is called **generic** if it does not have any parallel edges.

Stable Polygons

- We do a similar construction to build stable polygons, but our building blocks will be a little bit more sensitive so a certain amount of care must be taken.
- An *n*-gon is called **generic** if it does not have any parallel edges.
- Denote the subspace of generic polygons by M_r^0, which can be identified with $\mathcal{M}_{0,n}$ and so is an open complex analytic space

Stable Polygons

- We do a similar construction to build stable polygons, but our building blocks will be a little bit more sensitive so a certain amount of care must be taken.
- An n-gon is called **generic** if it does not have any parallel edges.
- Denote the subspace of generic polygons by M_r^0, which can be identified with $\mathcal{M}_{0,n}$ and so is an open complex analytic space
- Compactify M_r^0 by adding in appropriate limiting objects

Stable Polygons

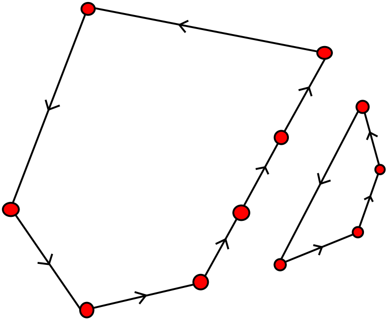
- We do a similar construction to build stable polygons, but our building blocks will be a little bit more sensitive so a certain amount of care must be taken.
- An *n*-gon is called **generic** if it does not have any parallel edges.
- Denote the subspace of generic polygons by M_r^0, which can be identified with $\mathcal{M}_{0,n}$ and so is an open complex analytic space
- Compactify M_r^0 by adding in appropriate limiting objects
- The basic idea is that whenever a set of edges become parallel, we resolve by introducing a "bubble" polygon

Stable Polygons

- Suppose a polygon P is a limit of a sequence of polygons in M_r^0 and P degenerates at a subset I of cardinality k

Stable Polygons

- Suppose a polygon P is a limit of a sequence of polygons in M_r^0 and P degenerates at a subset I of cardinality k
- Pair P with a generic $(k+1)$-gon P' whose first k-sides inherit the lengths of the original parallel edges of P but whose last side length is $r_{i_1} + r_{i_2} + \cdots + r_{i_k} - \epsilon_{i_1,\ldots,i_k}$.



Stable Polygons

- Different choices of these ϵ's will lead to equivalent complex analytic spaces, but will end up with different Kähler structures.

Stable Polygons

- Different choices of these ϵ's will lead to equivalent complex analytic spaces, but will end up with different Kähler structures.
- It will suffice to use the same ϵ in all cases, so we let $\epsilon = \min\{r_1, \ldots, r_n\}$.

Stable Polygons

- As a consequence of this choice, if we define

$$r_I = \left(r_{i_1}, \ldots, r_{i_k}, \sum_I r_i - \epsilon \right)$$

then r_I lies in a favorable chamber Δ_{k+1} in $\mathbb{D}_2^{k+1}$.

Stable Polygons

- As a consequence of this choice, if we define

$$r_I = \left(r_{i_1}, \ldots, r_{i_k}, \sum_I r_i - \epsilon \right)$$

then r_I lies in a favorable chamber Δ_{k+1} in $\mathbb{D}_2^{k+1}$.

- M_{r_I} is isomorphic to $\mathbb{CP}^{k-2}$ and no polygon in M_{r_I} degenerates at the $(k+1)^{\mathrm{st}}$ edge

- As a consequence of this choice, if we define

$$r_I = \left(r_{i_1}, \ldots, r_{i_k}, \sum_I r_i - \epsilon \right)$$

then r_I lies in a favorable chamber Δ_{k+1} in $\mathbb{D}_2^{k+1}$.
- M_{r_I} is isomorphic to $\mathbb{CP}^{k-2}$ and no polygon in M_{r_I} degenerates at the $(k+1)^{\text{st}}$ edge
- P' can degenerate at other edges, in which case we pair (P, P') with a third polygon P'' constructed similarly.

Stable Polygons

- As a consequence of this choice, if we define

$$r_I = \left(r_{i_1}, \ldots, r_{i_k}, \sum_I r_i - \epsilon\right)$$

  then r_I lies in a favorable chamber Δ_{k+1} in $\mathbb{D}_2^{k+1}$.
- M_{r_I} is isomorphic to $\mathbb{CP}^{k-2}$ and no polygon in M_{r_I} degenerates at the $(k+1)^{\mathrm{st}}$ edge
- P' can degenerate at other edges, in which case we pair (P, P') with a third polygon P'' constructed similarly.
- Our choice of ϵ guarantees that there are only finitely many degenerations that are possible, and hence this process terminates after a finite number of steps.

Stable Polygons

- A proper subset $I \subset \{1, \dots, n\}$ with $|I| \geq 2$ and r_I as defined above give a pair
$$(P, P') \in M_r \times M_{r_I}$$
called a **bubble pair** (We call P' a **bubble of** P if P degenerates at the edges e_I.)

Stable Polygons

- A proper subset $I \subset \{1, \ldots, n\}$ with $|I| \geq 2$ and r_I as defined above give a pair
$$(P, P') \in M_r \times M_{r_I}$$
called a **bubble pair** (We call P' a **bubble of** P if P degenerates at the edges e_I.)
- P' never degenerates at its longest edge

Stable Polygons

- A proper subset $I \subset \{1, \ldots, n\}$ with $|I| \geq 2$ and r_I as defined above give a pair
$$(P, P') \in M_r \times M_{r_I}$$
called a **bubble pair** (We call P' a **bubble of** P if P degenerates at the edges e_I.)
- P' never degenerates at its longest edge
- As a consequence of the triangle inequality, $(P, P') \in M_r \times M_{r_I}$ being a bubble pair implies
$$\sum_I r_i \leq \sum_{I^c} r_i. \tag{1}$$

Stable Polygons

- A proper subset $I \subset \{1, \ldots, n\}$ with $|I| \geq 2$ and r_I as defined above give a pair

$$(P, P') \in M_r \times M_{r_I}$$

  called a **bubble pair** (We call P' a **bubble of** P if P degenerates at the edges e_I.)
- P' never degenerates at its longest edge
- As a consequence of the triangle inequality, $(P, P') \in M_r \times M_{r_I}$ being a bubble pair implies

$$\sum_I r_i \leq \sum_{I^c} r_i. \tag{1}$$

- We say that a proper subset $I \subset 1, \ldots, n$ with $|I| \geq 2$ is a **relevant** subset if r satisfies (1). The collection of all relevant subsets of $\{1, \ldots, n\}$ will be denoted by $\mathcal{R}(r)$.

Stable Polygons

- A **stable** n**-gon** with respect to the side length vector r is a collection of labeled (but not ordered) polygons

$$\mathbf{P} := (P_0, P_1, \ldots, P_m) \in M_r \times M_{r_{I_1}} \times \cdots \times M_{r_{I_m}}$$

where the I_j's range over $\mathcal{R}(r)$

Stable Polygons

- A **stable** ***n*-gon** with respect to the side length vector r is a collection of labeled (but not ordered) polygons

$$\mathbf{P} := (P_0, P_1, \ldots, P_m) \in M_r \times M_{r_{I_1}} \times \cdots \times M_{r_{I_m}}$$

  where the I_j's range over $\mathcal{R}(r)$
- The following two properties are satisfied:

Stable Polygons

- A **stable** n-**gon** with respect to the side length vector r is a collection of labeled (but not ordered) polygons

$$\mathbf{P} := (P_0, P_1, \ldots, P_m) \in M_r \times M_{r_{I_1}} \times \cdots \times M_{r_{I_m}}$$

where the I_j's range over $\mathcal{R}(r)$

- The following two properties are satisfied:
  - Whenever $I_t \subset I_s$, then P_t is a bubble of P_s,

Stable Polygons

- A **stable** ***n*****-gon** with respect to the side length vector r is a collection of labeled (but not ordered) polygons

$$\mathbf{P} := (P_0, P_1, \ldots, P_m) \in M_r \times M_{r_{I_1}} \times \cdots \times M_{r_{I_m}}$$

  where the I_j's range over $\mathcal{R}(r)$
- The following two properties are satisfied:
  - Whenever $I_t \subset I_s$, then P_t is a bubble of P_s,
  - If P_h does not have a bubble, then it is generic (i.e. $P_h \in M^0_{r_{I_h}}$).

Stable Polygons

- A **stable** n**-gon** with respect to the side length vector r is a collection of labeled (but not ordered) polygons

$$\mathbf{P} := (P_0, P_1, \dots, P_m) \in M_r \times M_{r_{I_1}} \times \cdots \times M_{r_{I_m}}$$

  where the I_j's range over $\mathcal{R}(r)$
- The following two properties are satisfied:
  - Whenever $I_t \subset I_s$, then P_t is a bubble of P_s,
  - If P_h does not have a bubble, then it is generic (i.e. $P_h \in M^0_{r_{I_h}}$).
- The moduli space of stable polygons is denoted $\mathfrak{M}_{r,\epsilon}$.

Results

The main results of Hu that bring all of these ideas together:

Theorem

The moduli space $\mathfrak{M}_{r,\epsilon}$ carries a natural smooth, compact complex analytic structure.

Theorem

The moduli spaces $\mathfrak{M}_{r,\epsilon}$ and $\overline{\mathcal{M}}_{0,n}$ are biholomorphic. As a consequence, the complex structure of $\mathfrak{M}_{r,\epsilon}$ is independent of r and ϵ

the Kähler structure may depend on the choices of r and ϵ. It follows that one may use the moduli spaces of stable polygons to study the Kähler cone in $H^2(\overline{\mathcal{M}}_{0,n})$.

Results

Back to our original moduli spaces of polygons M_r:

Theorem

By forgetting all the bubbles, we obtain a natural projection

$$\pi_{r,\epsilon} : \mathfrak{M}_{r,\epsilon} \to M_r$$

that is holomorphic and bimeromorphic. It is the iterated blow up of M_r along (the proper transforms of) some explicitly described subvarieties Y_α when M_r is smooth. When M_r is singular, $\pi_{r,\epsilon} : \mathfrak{M}_{r,\epsilon} \to M_r$ is the composite of a canonical resolution of singularities followed by explicit iterated blowups.

This result allows us to view the moduli spaces M_r as a single family of smooth complex-analytic spaces $\mathfrak{M}_{r,\epsilon}$ whether M_r started out as singular or not.

Conclusion

- Start with something simple: a polygon in $\mathbb{R}^3$ with prescribed edge lengths

Conclusion

- Start with something simple: a polygon in $\mathbb{R}^3$ with prescribed edge lengths
- Consider the deformation space, allowing the angles to vary: we obtain a topological space that can be realized as a symplectic reduction or as a GIT quotient, illustrating the Kirwan-Kempf-Ness theorem.

Conclusion

- Start with something simple: a polygon in $\mathbb{R}^3$ with prescribed edge lengths
- Consider the deformation space, allowing the angles to vary: we obtain a topological space that can be realized as a symplectic reduction or as a GIT quotient, illustrating the Kirwan-Kempf-Ness theorem.
- Resolving singularities and blowing up along specified subvarieties yields a space isomorphic to the Deligne-Mumford compactification of the space of n-pointed genus zero algebraic curves.

Conclusion

- Start with something simple: a polygon in $\mathbb{R}^3$ with prescribed edge lengths
- Consider the deformation space, allowing the angles to vary: we obtain a topological space that can be realized as a symplectic reduction or as a GIT quotient, illustrating the Kirwan-Kempf-Ness theorem.
- Resolving singularities and blowing up along specified subvarieties yields a space isomorphic to the Deligne-Mumford compactification of the space of n-pointed genus zero algebraic curves.
- The moduli space of n-gons with a prescribed length is dominated by $\overline{\mathcal{M}_{0,n}}$.

Conclusion

Finally, we extend a thanks and wishes for a happy birthday to our advisor, Dr. Philip Foth.