# Consecutive Sums

Matt Thomas

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What patters did you find when investigating consecutive sums?

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## Proof.

Consider an odd number $N$. Both $N+1$ and $N-1$ are even, so $\frac{N+1}{2}$ and $\frac{N-1}{2}$ are both integers. $\frac{N+1}{2}$ and $\frac{N-1}{2}$ are consecutive since

$$\frac{N-1}{2}+1=\frac{N-1}{2}+\frac{2}{2}=\frac{N+1}{2}. \tag{2}$$

Finally,

$$\frac{N-1}{2}+\frac{N+1}{2}=\frac{2N}{2}=N, \tag{3}$$

so $\frac{N-1}{2}$ and $\frac{N+1}{2}$ are consecutive integers which sum to $N$. □

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## Fact

*The average of a collection of numbers is given by the sum divided by the size of the collection. Therefore, the sum of a collection of numbers is equal to the average times the number of terms.*

$$\text{sum} = \text{avg} \cdot (\text{number of terms})$$

# Odd Factors

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*If $N = td$ where $d$ is odd and greater than 1, $N$ can be written as a sum of consecutive integers.*

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$$\underbrace{\cdots + \frac{d-1}{2}}_{\text{t terms}} + \underbrace{\frac{d+1}{2} + \cdots}_{\text{t terms}}.$$

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- The number of terms is $2t$
- The product of these is $\frac{d}{2}2t = td = N$, as expected.

# Test

## Example

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## Example

$50 = 5*10$, so $d = 5$ and $t = 10$. The above algorithm produces $50 = -7 + -6 + -5 + \cdots + 7 + 8 + 9 + 10 + 11 + 12$. If we wish to consider only positive numbers, then we can notice that $-7 + -6 + \cdots + 6 + 7 = 0$, so $50 = 8 + 9 + 10 + 11 + 12$.

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- if the number has an odd factor, it can be written as a sum of consecutive integers
- We are only left with numbers which do not have any odd factors - $2^n$
- By a bit of experimentation, we hypothesize that these numbers cannot be written as a sum of consecutive integers

## Proposition

*If $N = 2^n$, then $N$ cannot be written as a sum of consecutive integers.*

## Proof.

Suppose $N$ can be written as a sum of consecutive integers, say $N = a + \cdots + b$. We will again use our fact.

- The average of $a$ and $b$ is $\frac{a+b}{2}$. The average of $a+1$ and $b-1$ is also $\frac{a+b}{2}$. The number of terms is $b - a + 1$. We now know that

$$N = a + \cdots + b = \frac{a+b}{2}(b - a + 1) = \frac{(a+b)(b-a+1)}{2}$$

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$$N = a + \cdots + b = \frac{a+b}{2}(b-a+1) = \frac{(a+b)(b-a+1)}{2}$$

- If both $a$ and $b$ are even, $b-a+1$ must be odd
- If both $a$ and $b$ are odd, then $b-a+1$ is odd
- If $a$ is even and $b$ is odd, $a+b$ is odd.
- If $a$ is odd and $b$ is even, $a+b$ is odd.

# Odd Factors

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- No matter what the pairing, one of the factors of $N$ is odd.
- This means that *if* the number $N$ can be written as a sum of consecutive integers, then it must have an odd factor. Since number of the form $2^n$ do not have any odd factors, they cannot be written as a sum of consecutive integers.

# The new question

Given a number $N$, determine the number of ways that $N$ can be written as a sum of consecutive integers.

We begin by determining which numbers can be written as a sum of 2 numbers, sum of 3 numbers, and so on. Consider the following table:

| | 2 numbers | 3 numbers | 4 numbers | 5 numbers |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | | | |
| 3 | 1+2 | | | |
| 4 | | | | |
| 5 | 2+3 | | | |
| 6 | | 1+2+3 | | |
| 7 | 3+4 | | | |
| 8 | | | | |
| 9 | 4+5 | 2+3+4 | | |
| 10 | | | 1+2+3+4 | |
| ⋮ | | | | |
| 15 | 7+8 | 4+5+6 | | 1+2+3+4+5 |

# A new formula

- the smallest number which can be written as a sum of $n$ number is $1 + 2 + \cdots + n$ (the triangular numbers)
- the formula for the $n^{th}$ triangular number is $\frac{(n)(n+1)}{2}$

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- for sum of 4 numbers: $10 + 4k$ where $k$ is an integer. The 4 is because we are in the 4th column, and the 10 because it's the 4th triangular number.
- The numbers which can be written as a sum of $n$ numbers is then given by $\frac{n(n+1)}{2} + nk$ where $k$ is an integer. We can easily write a computer program to check, then, whether numbers can be written as sums of consecutive integers.

## Example

50 can be written as a sum of 5 numbers since

$$\begin{aligned} 50 &= \frac{5 * 6}{2} + 5k \\ &= 15 + 5k \\ \implies 35 &= 5k \\ \implies 7 &= k, \end{aligned}$$

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which is an integer. 50 cannot be written as a sum of 3 numbers since

$$\begin{aligned} 50 &= \frac{3*4}{2} + 3k \\ &= 6 + 3k \\ \implies 44 &= 3k \\ \implies k &= \frac{44}{3}, \end{aligned}$$

which is not an integer.

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If a number shows up as a sum of an even number of positive integers, what can we say about another sum if we allow negatives?

# An even number of summands

Suppose $N$ can be written as a sum of an even number of consecutive integers. Let

$$N = (m - n + 1) + \cdots + m + \cdots + (m + n) \tag{6}$$

What can we now say?

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What can we now say?

- The average is $\frac{(m+n)+(m-n+1)}{2} = \frac{2m+1}{2}$.
- The number of terms is $(m + n) - (m - n + 1) + 1 = 2n$.

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- The average is $\frac{(m+n)+(m-n+1)}{2} = \frac{2m+1}{2}$.
- The number of terms is $(m + n) - (m - n + 1) + 1 = 2n$.
- The product, and thus $N$, is given by $(2m + 1)(n)$. Since $2m + 1$ is always odd, we can use this to help us.

# An odd number of summands

Now suppose the number of terms in the sum is odd. Say,

$$N = (m - n) + \cdots + m + (m + 1) + \cdots + (m + n). \qquad (7)$$

Again, we will use fact 2. The average is $\frac{(m+n)+(m-n)}{2} = m$. The number of terms is $(m + n) - (m - n) + 1 = 2n + 1$. $N$ is thus equal to $(m)(2n + 1)$. Again, we have one term, $2n + 1$, which must be odd.

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- which would make $m$ 0, 2, or 12 respectively,
- and thus $n$ is 50, 10, and 2 respectively.
- These correspond to sums $-49 + -48 + \cdots + 0 + 1 + 2 + 49 + 50$, $-7 + -6 + \cdots + 2 + 3 + \cdots + 12$, and $11 + 12 + 13 + 14$.

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- and thus $n$ is 50, 10, and 2 respectively.
- These correspond to sums $-49 + -48 + \cdots + 0 + 1 + 2 + 49 + 50$, $-7 + -6 + \cdots + 2 + 3 + \cdots + 12$, and $11 + 12 + 13 + 14$.
- Note the first sum, when only considering positive terms, is just 50, so this is a trivial case. It is always true that $m = 0$ will correspond to this trivial case. The second sum is equivalent to $8 + 9 + 10 + 11 + 12$.

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- $n$ is then 0, 2, or 12.
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- $n$ is then 0, 2, or 12.
- $m$ is then 50, 10, or 2.
- These correspond to the sums 50, $8+9+10+11+12$, or $-10+-9+\cdots+2+\cdots 13+14$.

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## Example

- If 50 is written as a sum of an odd number of terms, $2n + 1$ could be equal to 1, 5, or 25.
- $n$ is then 0, 2, or 12.
- $m$ is then 50, 10, or 2.
- These correspond to the sums 50, $8 + 9 + 10 + 11 + 12$, or $-10 + -9 + \cdots + 2 + \cdots 13 + 14$.
- This last sum is equivalent to $11 + 12 + 13 + 14$. This shows all of the ways that 50 can be written as a sum of consecutive integers.

# Another Example

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$N = 45 = 3^2 * 5.$

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$N = 45 = 3^2 * 5.$

All the odd factors excluding 1 are 3, 5, 9, 15, and 45, so there are 5 ways to write 45 as a sum of positive consecutive integers

# Observations

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- Note that since only odd factors affect the number of ways $N$ can be written, factors of 2 never affect the number of ways $N$ can be written.
- In particular, any number and double that number (or half of it, if it is even) will have the same number of decompositions.

**Theorem**

*Suppose $N$ is written in its prime decomposition as $N = 2^{n_0} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$. The 2's, as mentioned, are irrelevant. There are $(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)$ possible odd factors, which includes 1, so $N$ can be written as a sum of consecutive positive integers in $[(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)] - 1$ distinct ways.*

# The last example

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$21168 = 2^4 \cdot 3^3 \cdot 7^2$ can be written as a sum of positive consecutive integers in $(3+1)(2+1) - 1 = 4 \cdot 3 - 1 = 12 - 1 = 11$ distinct ways.