TWO IDEAS FROM INTERSECTION THEORY

ALEX PETROV

This is an expository paper based on Serre's *Local Algebra* (denoted throughout by [**Ser**]). The goal is to describe simple cases of two powerful ideas in intersection theory: (1) one called *reduction to the diagonal* is introduced in Section 3; (2) the other is Serre's definition of *intersection multiplicity* (which takes up Section 4). Before that we give a quick section on how algebra and geometry are related (at least in the affine, classical case), this is Section 1; and we prove Bezout's theorem for curves (arguably the most famous result in intersection theory), the theorem takes up Section 2.

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1. A Short Algebro-Geometric Lexicon

Let k be an algebraically closed field.

A subset $V \subset \mathbb{A}^n(k)$ is called an *affine algebraic set* if there are polynomials $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$ such that V is the solution set of the system of equations

$$f_i(X_1, \ldots, X_n) = 0 \qquad\qquad i = 1, \ldots, m.$$

For any algebraic set V we define

$$\mathcal{I}_V = \{f \in k[X_1, \ldots, X_n] : f(V) = 0\}.$$

This is an ideal of $k[X_1, \ldots, X_n]$. An algebraic set V for which $\mathcal{I}_V$ is prime is called an *affine algebraic variety*. The *coordinate ring* of an algebraic variety V is the domain

$$k[V] = k[X_1, \ldots, X_n]/\mathcal{I}_V.$$

We record some facts about varieties that we will need.

(1) An affine algebraic set V is irreducible if and only if $\mathcal{I}_V$ is prime, i.e. varieties are irreducible.

(2) W is an irreducible subvariety of a variety V if and only if $\mathcal{I}_W$ is prime in $k[V]$.
(3) The Cartesian product $V \times U \subset k[X_1, \ldots, X_n, Y_1, \ldots, Y_l]$ of two varieties V, U is an algebraic set. Moreover we have

$$k[V \times U] \cong k[V] \otimes_k k[U].$$

2. The Case of Curves

An algebraic set $V \subset \mathbb{P}^2(k)$ is called a *projective curve* iff there is a homogeneous polynomial $F \in k[X_1, X_2, X_3]$ such that $\mathcal{V}(F) = V$. We will denote V by F.

Let F and G be two projective curves. For any $P \in \mathbb{P}^2(k)$. Let

$$\mathcal{O}_P = \left\{ \frac{\phi}{\psi} : \phi, \psi \text{ are homogeneous of the same degree }, \psi(P) \neq 0 \right\}$$

be the *local ring at* P. This is indeed a local ring. If $P = (a, b)$ is a point in an affine piece then $\mathcal{O}_P \cong k[X, Y]_{(X-a, Y-b)}$.

Define

$$F_P = F \cdot \mathcal{O}_P, \qquad G_P = G \cdot \mathcal{O}_P, \qquad \mathcal{O}_{F \cap G, P} = \mathcal{O}_P / (F_P + G_P).$$

The *intersection multiplicity* of F and G at P is defined to be

$$\mu_P(F, G) = \mathtt{dim}_k \mathcal{O}_{F \cap G, P}.$$

We will see that this number is indeed finite in the proof of Bezout's theorem.

Alternatively, notice that since F and G do not share components they must intersect in finitely many points. Therefore the ring $A = k[X, Y]/(f, g)$ has a finite spectrum that is equal to its maximal spectrum, that is $\mathtt{Spec}(A) = \mathtt{Max}(A)$ and $|\mathtt{Spec}(A)| < \infty$. In the language of commutative algebra this says that A is an Artin ring (since it is Noetherian and has dimension 0). Let $\mathtt{Max(A)} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$ and consider $A/\mathfrak{m}_i = K_i$. By the Weak Nullstellensatz this is an algebraic extension of k, so $\mathtt{dim}_k K_i < \infty$ (both results use the fact that A is finitely generated over k). From the proof of the structure theorem of Artin rings (p. 90 of [**A-M**]) we know that there for some integer $t > 0$ we have

$$A = \prod_{i=1}^{r} A/\mathfrak{m}_i^t.$$

Hence we must have

$$\mathtt{dim}_k A = \sum_{i=1}^{r} \mathtt{dim}_k (A/\mathfrak{m}_i^t).$$

But $\mathtt{dim}_k(A/\mathfrak{m}_i) < \infty \implies \mathtt{dim}_k(A/\mathfrak{m}_i^t) < \infty$ by induction. In fact, we have

$$A/\mathfrak{m}_i \cong (A/\mathfrak{m}_i^2)/(\mathfrak{m}_i/\mathfrak{m}_i^2)$$

and the map $A/\mathfrak{m}_i \to \mathfrak{m}_i/\mathfrak{m}_i^2$, $m \mapsto x \cdot m$ is onto for any $x \in \mathfrak{m}_i$. So $A/\mathfrak{m}_i^2$ has finite dimension over k. Similarly, by induction we get that $\mathtt{dim}_k(A/\mathfrak{m}_i^t)$ for any integer $t > 0$. This proves that

$$\mathtt{dim}_k A < \infty$$

by the formula above. Now the localization will also have finite dimension over k.

A *cycle* of $\mathbb{P}^2$ is an element of the free abelian group of points

$$Z = \sum_{P \in \mathbb{P}^2} m_P \cdot P.$$

The *degree* of Z is $\mathtt{deg} Z = \sum_P m_P$. Notice that a cycle in this notation is a geometric object (as a set of point with given weights) and an algebraic object (as a set of prime ideals with given weights).

The intersection of two curves with no common components is then the cycle

$$F \star G = \sum_{P \in \mathbb{P}^2} \mu_P(F, G) \cdot P.$$

To motivate the definition (and to show that this is indeed a cycle) one must prove the following properties of $\mu_P(F, G)$.

Theorem: If F and G do not intersect at P iff $\mu_P(F, G) = 0$.

Proof: WLOG assume that $P = [0 : 0 : 1]$. This can always be achieved by changing variables. Then we can view P as the origin in affine space and by our convention above we have

$$\mathcal{O}_P = k[X, Y]_{(X,Y)} = \left\{ \frac{f}{g} : f, g \in k[X, Y], g(0, 0) \neq 0 \right\}.$$

If F does not pass through P then $F(0, 0) \neq 0$ and therefore F_P contains the unit F. It follows that $F_P + G_P = \mathcal{O}_P$ and hence the intersection multiplicity is 0. The same argument applies to the case when $G(0, 0) \neq 0$.

On the other hand if $\mathcal{O}_{F \cap G, P} = 0$ then $\mathcal{O}_P = F_P + G_P$. Hence we have

$$1 = \frac{\phi_1}{\psi_1} + \frac{\phi_2}{\psi_2} = \frac{\chi}{\psi_1 \psi_2},$$

where $\chi \in F + G$. This shows that $P \notin \mathcal{V}(F) + \mathcal{V}(G) = F \cap G$. And the proof is complete $\square$

Bezout's Theorem: If F and G do not share irreducible components then have the following formula

$$\mathtt{deg}(F \star G) = \mathtt{deg}F \cdot \mathtt{deg}G.$$

Proof: Let $P_1, \ldots, P_t$ be the points of intersection of the two curves. WLOG assume that they all lie in the affine piece where $P_i = [a_i : b_i : 1]$. Then we can look at the affine varieties $f = F \cap \mathbb{A}^2(k)$, $g = G \cap \mathbb{A}^2(k)$.

The ring $k[f \cap g]$ has only finitely many prime ideals (all of which are maximal). More precisely these are the ideals $\mathfrak{m}_i = (X - a_i, Y - b_i)$. In the language of commutative algebra we say that $k[f \cap g]$ is a *semi-local* ring (or *Artin* ring if it is also Noetherian). We need a lemma.

Lemma: Let A be a semi-local ring so that $\mathtt{Spec}(A) = \mathtt{Max}(A) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$. Then $A \cong A_{\mathfrak{m}_1} \times \cdots \times A_{\mathfrak{m}_r}$.

To prove this result start by defining $\mathfrak{q}_i = \mathtt{ker}(A \to A_{\mathfrak{m}_i})$ (this is the canonical map). The map is actually onto because localization and quotients commute

$$A/\mathfrak{q}_i \cong (A/\mathfrak{q}_i)_{(\mathfrak{m}_i/\mathfrak{q}_i)} \cong A_{\mathfrak{m}_i}/\mathfrak{q}_i A_{\mathfrak{m}_i} \cong A_{\mathfrak{m}_i}.$$

The middle isomorphism comes from the fact that localization and quotients commute. The first isomorphism comes from the fact that all the elements of $A/\mathfrak{q}_i - \mathfrak{m}_i/\mathfrak{q}_i$ are units of $A/\mathfrak{q}_i$. (To see this let $\alpha + \mathfrak{q}_i \in A/\mathfrak{q}_i - \mathfrak{m}_i/\mathfrak{q}_i$ then $\alpha \in A - \mathfrak{m}_i$, so α is a unit in $A_{\mathfrak{m}_i}$ and by the second isomorphism it follows that $\alpha + \mathfrak{q}_i$ is a unit in $A/\mathfrak{q}_i$.) The last isomorphism comes from the fact that $\mathfrak{q}_i A_{\mathfrak{m}_i}$ is the zero ideal of $A_{\mathfrak{m}_i}$ by definition of $\mathfrak{q}_i$.

Next, we show that the $\mathfrak{q}_i$'s are relatively prime. To that end notice that $A_{\mathfrak{m}_i}$ is a local ring and the ideal $\mathfrak{m}_i A_{\mathfrak{m}_i}$ is the Jacobson radical, i.e. consists only of nilpotent elements. Then the ideal $\mathfrak{m}_i/\mathfrak{q}_i \subset A/\mathfrak{q}_i$ consists only of nilpotent element. In particular, for any $x \in \mathfrak{m}_i$ there is d so that $x^d \in \mathfrak{q}_i$.

Suppose that $\mathfrak{q}_i \subset \mathfrak{m}_j$, then $x^d \in \mathfrak{m}_j$ for each $x \in \mathfrak{m}_i$ and therefore by $\mathfrak{m}_i = \mathfrak{m}_j$. This shows that the $\mathfrak{q}_i$'s are relatively prime.

We show further that $\cap^n \mathfrak{q}_i = 0$. Let $x \in \cap^n \mathfrak{q}_i$. Then for each j we have $x/1 = 0$ in $A_{\mathfrak{m}_j}$, i.e. there exists $g_j \in A - \mathfrak{m}_j$ so that $g_j x = 0$. Look at $I = (g_1, \ldots, g_r)$. We see that $Ix = 0$, but $I \not\subset \mathfrak{m}_j$ for any j so $I = A$. Then $1 \in I$ and we get $x = 1x = 0$.

Using the Chinese Remainder Theorem we get

$$A \cong A/\mathfrak{q}_1 \times \cdots \times A/\mathfrak{q}_r \cong A_{\mathfrak{m}_1} \times \cdots \times A_{\mathfrak{m}_r} \ \S$$

Applying the previous lemma to $k[f \cap g]$ gives

$$k[f \cap g] \cong k[f \cap g]_{\mathfrak{m}_1} \times \cdots \times k[f \cap g]_{\mathfrak{m}_t}.$$

Notice that this shows that $\mu_P(F,G) = \mathtt{dim}_k k[f \cap g]_P$ is finite, since $k[f \cap g]$ is finitely generated k-algebra and $k[f \cap g]_P$ is a subspace of it (by the above lemma).

We compute the dimensions on both sides:

$$\mathtt{dim}_k k[f \cap g] = \sum_{i=1}^{t} \mu_{P_i}(F,G) = \mathtt{deg}(F \star G).$$

At this point we are faced with showing that $\mathtt{dim}_k k[f \cap g] = \mathtt{deg}(F) \cdot \mathtt{deg}(G)$. The argument that we give follows Fulton (see [**Ful1**]).

Let $k[X,Y,Z] = A$ and let $\Gamma^* = k[X,Y]/(f,g)$ and $\Gamma = k[X,Y,Z]/(F,G)$ and let Γ_d (respectively A_d) be the vector space of degree d form in Γ (respectively A). Set $n = \mathtt{deg}F = \mathtt{deg}f$ and $m = \mathtt{deg}G = \mathtt{deg}g$. We want to show that $\mathtt{dim}_k \Gamma^* = \mathtt{dim}_k \Gamma_d = mn$ for some d.

Step 1: $\mathtt{dim}\Gamma_d = mn$ for all $d \geq m + n$.

Let $\pi : A \to \Gamma$ be the natural projection, let $\phi : A \times A \to A$ be defined by $\phi(S,T) = SF + TG$, and let $\psi : A \to A \times A$ be defined by $\psi(Q) = (GQ, -FQ)$.

Then we have the following facts: (1) $\mathtt{ker}\psi = 0$, because $0 = (GQ, -FQ) \implies Q = 0$; (2) $\mathtt{ker}\phi = \mathtt{Im}\psi$, indeed if $(S,T) \in \mathtt{Im}\psi$ then $S = GQ, T = -FQ$ so $\phi(S,T) = FGQ - FGQ = 0$ and if $FS + GT = 0$ then $FS = -GT$, but F and G share no components so we must have $S = GQ$ and $T = -FQ$ for some Q; (3) It is clear from the definitions that $\mathtt{ker}\pi = \mathtt{Im}\phi$ and that π surjects. Summarizing all of these amounts to saying that we have an exact sequence

$$0 \longrightarrow A \xrightarrow{\psi} A \times A \xrightarrow{\phi} A \xrightarrow{\pi} \Gamma \longrightarrow 0.$$

By restriction to various degrees we get exact sequences:

$$0 \longrightarrow A_{d-m-n} \xrightarrow{\psi} A_{d-m} \times A_{d-n} \xrightarrow{\phi} A_d \xrightarrow{\pi} \Gamma_d \to 0.$$

One can prove by induction that $\mathtt{dim}A_d = \frac{(d+1)(d+2)}{2}$ and so if $d \geq m+n$ we have from the exact sequence

$$1 + \frac{(d+1)(d+2)}{2} = \mathtt{dim}\Gamma_d + \frac{(d-m+1)(d-m+1)}{2} + \frac{(d-n+1)(d-n+2)}{2}$$
$$\implies \mathtt{dim}\Gamma_d = mn.$$

Step 2: The map $\alpha : \Gamma \to \Gamma$ given by $\alpha(\overline{H}) = \overline{Z \cdot H}$, where Z is the Z-coordinate is one-to-one. This is just a computation and the reader can see it in [**Ful1**].

Step 3: Let $d \geq m+n$, and choose $Q_1, \ldots, Q_{mn} \in A_d$ whose residues in Γ_d form a basis for Γ_d. Let $q_i \in k[X,Y]$ be the dehomogenization of Q_i and let a_i be the residue of q_i in Γ^*. Then $a_1, \ldots, a_{mn}$ form a basis for Γ^*.

The map from Step 2 is one-to-one, but Γ_d and Γ_{d+1} have the same dimension for $d \geq m+n$, so it restricts to an isomorphism from Γ_d to Γ_{d+1}. Thus the residues of $Z^r Q_1, \ldots, Z^r Q_{mn}$ form a basis for Γ_{d+r} for all $r \geq 0$.

The a_i's generate Γ^*: if $h = \overline{h} \in \Gamma^*$, $H \in K[X,Y]$ some $Z^N H$ is a form of degree $d+r$, so

$$Z^N H = \sum_{i=1}^{mn} \lambda_i Z^r Q_i + BF + CG$$

then $h = \sum \lambda_i a_i$.

The a_i are independent: if $\sum \lambda_i a_i = 0$ lifting to Γ_{d+r} shows that there

$$\sum \lambda_i \overline{Z^r A_i} = 0 \implies \lambda_i = 0 \text{ for all } i.$$

This proves that for some $d \geq m+n$

$$\mathtt{dim}\Gamma_d = \mathtt{dim}\Gamma^* = \mathtt{dim}_k k[f \cap g] = mn = \mathtt{deg}F \cdot \mathtt{deg}G$$

and accomplishes the proof of Bezout's theorem $\square$.

Mathematicians have developed various generalizations of the following ideas to higher dimensions. For the rest of this paper we will look at two ideas that have played a major role in this development. First, we look at an idea called "reduction to the diagonal" that reduces the question of intersection of arbitrary affine varieties to a question about intersection of a variety with a variety of a special kind (a linear variety). Next, we look at a generalization of the local multiplicity numbers due to Serre.

3. Reduction to the Diagonal

For an affine variety $V \subset \mathbb{A}^n(k)$ we define $\mathtt{dim}V$ to be the Krull dimension of the Noetherian domain $k[V]$.

Let A be a Noetherian ring. We define the *height* $\mathtt{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ to be the Krull dimension of the ring $A_{\mathfrak{p}}$. Notice that this is the supremum of lengths of prime ideal chains that end with $\mathfrak{p}$. For a general ideal $\mathfrak{a}$ we define its height $\texttt{ht}(a)$ to be the infimum of heights of prime ideals contained in $\mathcal{V}(\mathfrak{a})$. Formulaically we have defined

$$\texttt{ht}(\mathfrak{p}) = \texttt{dim}A_{\mathfrak{p}}, \qquad \texttt{ht}(\mathfrak{a}) = \inf_{\mathfrak{p}\in\mathcal{V}(\mathfrak{a})} \texttt{ht}(\mathfrak{p}).$$

An important fact that we will use again and again is that if we have A/B integral then by the Going Up theorem $\texttt{dim}A = \texttt{dim}B$.

Let V and U be two affine varieties. The set $V \cap U$ is not necessarily a variety, but it is an algebraic set. Let W be an irreducible component of $V \cap U$. We aim to show that

$$\texttt{dim}W \geq \texttt{dim}V + \texttt{dim}U - n.$$

If we reformulate this in the language of commutative algebra we see that the result states:

Theorem: If $\mathfrak{p}'$ and $\mathfrak{p}''$ are two prime ideals of the polynomial algebra $A = k[X_1, \ldots, X_n]$. If $\mathfrak{p}$ is a minimal element of $\mathcal{V}(\mathfrak{p}' + \mathfrak{p}'')$, we have

$$\texttt{ht}(\mathfrak{p}) \leq \texttt{ht}(\mathfrak{p}') + \texttt{ht}(\mathfrak{p}'').$$

Proof: The following proof is taken from Serre. The idea is instead of looking at $A, \mathfrak{p}'$ and $\mathfrak{p}''$ to look at $A \otimes_k A$ and two prime ideals that correspond to $V \times W$ and the diagonal $\triangle$.

We need a couple of lemmas:

Lemma: Let A' and A'' be domains which are finitely generated algebras over k. For every minimal prime ideal $\mathfrak{P}$ of $A' \otimes_k A''$, we have:

$$\texttt{dim}(A' \otimes_k A''/\mathfrak{P}) = \texttt{dim}(A' \otimes_k A'') = \texttt{dim}A' + \texttt{dim}A''.$$

To prove the lemma start with A' and A''. By Noether normalization there exist polynomial k-algebras $B' = k[x_1, \ldots, x_n]$ and $B'' = k[y_1, \ldots, y_m]$ such that A'/B' and A''/B'' are both integral. Therefore $\texttt{dim}A' = \texttt{dim}B'$ and $\texttt{dim}A'' = \texttt{dim}B''$.

Now consider $k[x_1, \ldots, x_n] \otimes_k k[y_1, \ldots, y_m]$. We can write each element of this tensor product as a k-linear combination of various $x_1, \ldots, x_n$ tensor $y_1, \ldots, y_m$ powers, so this is isomorphic to $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$. We know what the dimension of a polynomial algebra is, so we get $\texttt{dim}B' + \texttt{dim}B'' = \texttt{dim}(B' \otimes_k B'')$.

Now if A'/B' is integral then A' is finitely generated over B'. Similarly for A''/B''. But then $A' \otimes_k A''$ is finitely generated over $B' \otimes_k B''$ by the tensor products of the generators of A'/B' and A''/B''. Notice that we need the ring structure of $A' \otimes_k A''$ that we get from k-algebras for this result. So $A' \otimes_k A''$ is finitely generated over $B' \otimes_k B''$, hence integral. This shows that $\mathtt{dim}(A' \otimes_k A'') = \mathtt{dim}(B' \otimes_k B'')$.

So far we have shown that

$$\mathtt{dim}A' + \mathtt{dim}A'' = \mathtt{dim}B' + \mathtt{dim}B'' = \mathtt{dim}(B' \otimes_k B'') = \mathtt{dim}(A' \otimes_k A'').$$

Next let $\mathfrak{P}$ be a minimal prime of $A' \otimes A''$. Let B' and B'' be as before and let K', K'', L', L'' be the rings of fractions of A', A'', B', B'', respectively. We have the following diagram of inclusions

$$\begin{array}{ccc} L' \otimes_k L'' & \longrightarrow & K' \otimes_k K'' \\ \uparrow & & \uparrow \\ B' \otimes_k B'' & \longrightarrow & A' \otimes_k A'' \end{array}$$

K' is an L'-module, i.e. a vector space, so it is free over L'. Similarly for K'' and L''. But then because $\otimes$ and $\oplus$ commute we have that $K' \otimes_k K''$ is a free $L' \otimes_k L''$-module. Hence a torsion free $B' \otimes_k B''$-module.

Let $x \in \mathfrak{P} \cap (B' \otimes_k B'')$. Notice that we can think of x as an element of $A' \otimes_k A''$ and $B' \otimes B''$ at the same time. As an element of $A' \otimes_k A''$ we know that it is a zero divisor (because it is in $\mathfrak{P}$ and this ideal belongs to 0). It follows that there is $y \in A' \otimes_k A''$ such that $xy = 0$, so y is a torsion element. Therefore $x = 0$. We have just shown that $0 = \mathfrak{P} \cap B' \otimes_k B''$.

Now $A' \otimes_k A''/\mathfrak{P}$ is integral over $B' \otimes B''$ and we finally get

$$\mathtt{dim}(A' \otimes_k A''/\mathfrak{P}) = \mathtt{dim}(B' \otimes B''),$$

and this finishes the proof of the lemma. §

Lemma: Let A be a k-algebra, let $C = A \otimes_k A$, and let $\phi : C \to A$ be the homomorphism defined by $\phi(a \otimes b) = ab$.

(i) The kernel $\mathfrak{d}$ of ϕ is the ideal of C generated by the elements

$$1 \otimes a - a \otimes 1, \qquad \text{for } a \in A.$$

(ii) If $\mathfrak{p}'$ and $\mathfrak{p}''$ are two ideals of A, the image by ϕ of the ideal

$$\mathfrak{p}' \otimes A + A \otimes \mathfrak{p}''$$

is equal to $\mathfrak{p}' + \mathfrak{p}''$.

For a proof see [**Ser**].

Now we return to the proof of the theorem.

Consider the exact sequence:

$$0 \longrightarrow \mathfrak{p}' \otimes A + A \otimes \mathfrak{p}'' \longrightarrow A \otimes_k A \xrightarrow[v]{} A/\mathfrak{p}' \otimes_k A/\mathfrak{p}'' \longrightarrow 0.$$

We use the map $\phi : A \otimes_k A \to A$, $\phi(a \otimes b) = ab$. Let $\mathfrak{P} = \phi^{-1}(\mathfrak{p})$. Notice that $\mathfrak{P} \in \mathcal{V}(\mathfrak{d})$ and $\mathfrak{P} \in \mathcal{V}(\mathfrak{p}' \otimes A + A \otimes \mathfrak{p}'')$. Call $\mathfrak{r} = \mathfrak{p}' \otimes A + A \otimes \mathfrak{p}''$. Therefore $\mathfrak{P}$ is an element of $\mathcal{V}(\mathfrak{d}+\mathfrak{r})$. Moreover $\mathfrak{P}$ is a minimal element of $\mathcal{V}(\mathfrak{d} + \mathfrak{r})$.

Let $\mathfrak{Q}$ be the image of $\mathfrak{P}$ under v, $\mathfrak{d}'$ be the image of $\mathfrak{d}$ under v.

We can summarize these definitions with the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \mathfrak{r} \longrightarrow & \mathfrak{d} \subset \mathfrak{P} & \xrightarrow[v]{} & \mathfrak{d}' \subset \mathfrak{Q} \longrightarrow 0 \\ & \phi \big\downarrow & & \\ & 0 \subset \mathfrak{p} & & \end{array}$$

By the second lemma $\mathfrak{d}$ is generated by the n elements $X_i \otimes 1 - 1 \otimes X_i$ hence $\mathtt{ht}(\mathfrak{Q}) \leq n$ because $\mathfrak{Q}$ is a minimal element of $\mathcal{V}(\mathfrak{d} + \mathfrak{r})$. Let $\mathfrak{Q}_0$ be a minimal prime ideal contained in $\mathfrak{Q}$. We have $\mathtt{ht}(\mathfrak{Q}/\mathfrak{Q}_0) \leq n$. But according to the first lemma, we have

$$\mathtt{dim}(A/\mathfrak{p}' \otimes_k A/\mathfrak{p}'') = \mathtt{dim}(A/\mathfrak{p}') + \mathtt{dim}(A/\mathfrak{p}'');$$

since

$$\mathtt{ht}(\mathfrak{Q}/\mathfrak{Q}_0) = \mathtt{dim}(A/\mathfrak{p}' \otimes_k A/\mathfrak{p}''/\mathfrak{Q}_0) - \mathtt{dim}(A/\mathfrak{p}' \otimes_k A/\mathfrak{p}''/\mathfrak{Q}).$$

Combining the equations above with the fact that $\mathtt{dim}(D/\mathfrak{Q}) = \mathtt{dim}(A/\mathfrak{p})$ we get

$$n \geq \mathtt{ht}(\mathfrak{Q}/\mathfrak{Q}_0) = \mathtt{dim}(A/\mathfrak{p}') + \mathtt{dim}(A/\mathfrak{p}'') - \mathtt{dim}(A/\mathfrak{p})$$

hence

$$n - \mathtt{dim}(A/\mathfrak{p}) \leq n - \mathtt{dim}(A/\mathfrak{p}') + n - \mathtt{dim}(A/\mathfrak{p}'').$$

i.e.

$$\mathtt{ht}(\mathfrak{p}) \leq \mathtt{ht}(\mathfrak{p}') + \mathtt{ht}(\mathfrak{p}'').$$

This is the result that we want $\square$

We pause for a second and record what the commutative algebra above says geometrically.

The first lemma shows that every irreducible component of a product of two affine varieties of dimensions r and s has dimension $r + s$.

The significance of the theorem itself has already been discussed. The proof replaces the triple $(A, \mathfrak{p}', \mathfrak{p}'')$ with the triple $(A \otimes_k A, \mathfrak{d}, \mathfrak{r})$. This is called *reduction to the diagonal.* The geometric idea is instead of looking at $V \cap U$ to look at $(V \times U) \cap \triangle$, where $\triangle$ is the diagonal. Notice that this reduces the question of intersection of two arbitrary varieties to the question of intersection of a variety with a linear variety.

4. Serre's Definition of Intersection Multiplicity

Looking back at Bezout's Theorem for curves we see that it is crucial to generalize the concept of intersection multiplicity. We will present a generalization due to Serre.

Let V, U be two irreducible varieties and let W be an irreducible component of $V \cap U$. We assume that the ring

$$\mathcal{O}_W = \left\{ \frac{f}{g} : f, g \text{ homogeneous of the same degree }, g(W) \neq 0 \right\}$$

is a regular local ring.

To simplify notation let $\mathcal{O}_W = A$, the ideals corresponding to V and U are respectively $\mathfrak{p}_V$, $\mathfrak{p}_U$. Serre's definition of *intersection multiplicity* at W (also called *Euler-Poincare characteristic*) is:

$$\chi^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U) = \sum_{i=0}^{n} (-1)^i \mathtt{length}_A(\mathtt{Tor}_i^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U)).$$

To make sense of the definition and indeed to show that the Euler-Poincare characteristic is well-defined we must recast the definition of $\mathtt{Tor}_i^A$ in our specific case.

$(\mathtt{Tor}_i^A)$: Let A be a commutative ring and consider an exact sequence of A-modules

$$0 \longrightarrow B \longrightarrow C \qquad D \longrightarrow 0.$$

Tensoring the sequence with an A-module M (on the right) we obtain an exact sequence

$$B \otimes_A M \longrightarrow C \otimes_A M \longrightarrow D \otimes_A M \longrightarrow 0.$$

The kernel of the map on the left is not necessarily zero (if it were then M will be *flat*). Define $\mathtt{Tor}_1^A(D, M)$ to be the kernel of the leftmost map. Then we have an exact sequence

$$\mathtt{Tor}_1^A(D, M) \longrightarrow B \otimes_A M \longrightarrow C \otimes_A M \longrightarrow D \otimes_A M \longrightarrow 0.$$

Now again we do not know if the map on the left has zero kernel. So let $\mathtt{Tor}_1^A(C, M)$ be its kernel. One can continue this process indefinitely, setting $\mathtt{Tor}_0^A(\_,\_) = \_ \otimes_A \_$ we obtain an infinite exact sequence

$$\cdots \longrightarrow \mathtt{Tor}_{n+1}^A(D,M) \longrightarrow \mathtt{Tor}_n^A(B,M) \longrightarrow \mathtt{Tor}_n^A(C,M)$$

$$\longrightarrow \mathtt{Tor}_n^A(D,M) \longrightarrow \mathtt{Tor}_{n-1}^A(B,M) \longrightarrow \mathtt{Tor}_{n-1}^A(C,M)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cdots \longrightarrow \mathtt{Tor}_1^A(D,M \longrightarrow B \otimes_A M \longrightarrow C \otimes_A M$$

$$\longrightarrow D \otimes_A M \longrightarrow 0.$$

Properties of Tor that can be found is a homological algebra text now have direct bearing on intersection multiplicity. In our case we take the exact sequence of k-modules

$$0 \longrightarrow \mathfrak{p}_V \longrightarrow A \longrightarrow A/\mathfrak{p}_V \longrightarrow 0$$

and $M = A/\mathfrak{p}_U$.

Assume that the Euler-Poincare characteristic is well-defined, then:

Theorem: With the notation from the beginning of the section we have

$$\chi^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U) = \chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V).$$

Proof: In fact we have for all $n \geq 0$ that $\mathtt{Tor}_n^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U) = \mathtt{Tor}_n^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)$. This result can be found in any book on homological algebra and we have in fact proved the case when $n = 0$ in our commutative algebra class □

Note that we did not even mention this fact when we defined intersection multiplicity for curves. There (i.e. in Section 1) it is obvious that $\mu_P(F,G) = \mu_P(G,F)$.

Arguing that $\chi_A(A/\mathfrak{p}_V, A/\mathfrak{p}_U)$ is well-defined also requires some machinery from homological algebra.

Theorem: With our conventions we have that for all $n \geq 0$ the A-module $\mathtt{Tor}_n^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U)$ has finite length.

Proof: See page 106 (and the preceding ten pages) in [**Ser**]. □

Theorem: $\chi^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U)$ is a non-negative integer.
Proof: For a proof see [**Ser**], page 106. $\square$

These simple properties should convince the reader that Serre was on the right track when he made his definitions. In fact, we have:

Theorem: When $V = F, U = G$ are curves with the assumptions from Bezout's Theorem , $P = W \subset V \cap U$, $A = \mathcal{O}_P$ we have

$$\mu_P(F,G) = \chi^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U).$$

Proof: We begin by uniformizing our notational conventions. Remember that

$$\mu_P(F,G) = \mathtt{dim}_k \left(\frac{\mathcal{O}_P}{F_P + G_P} \right)$$

but $A = \mathcal{O}_P$, $F_P = \mathfrak{p}_V$, $G_P = \mathfrak{p}_U$, so in fact we have

$$\mu_P(F,G) = \mathtt{dim}_k A/(\mathfrak{p}_V + \mathfrak{p}_U).$$

Before we proceed we need to discuss the Koszul complex (in a simple case according to Serre).

Koszul Complex: Let A be a commutative ring, $x \in A$. Define the complex $K(x)$ by

$$\begin{array}{ccccccccc}
K_i(x) & \longrightarrow & \cdots & \longrightarrow & K_2(x) & \longrightarrow & K_1(x) & \xrightarrow{a \mapsto x \cdot a} & K_0(x) \\
\| & & & & \| & & \| & & \| \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{a \mapsto x \cdot a} & A
\end{array}$$

Choose a basis for A, so that the derivation $d : K_1(x) \to K_0(x)$ is given by $e_x \mapsto x$. Next, let M be a finitely generated A-module and consider the complex $K(x) \otimes_A M$, which we denote by $K(x, M)$. Explicitly this produces the complex

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & K_2(x) \otimes_A M & \longrightarrow & K_1(x) \otimes_A M & \xrightarrow{d} & K_0(x) \otimes_A M \\
& & \| & & \| & & \| \\
\cdots & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{d} & M
\end{array}$$

The derivation now is $d(e_x \otimes m) = xm$. The homology modules of $K(x, M)$ are

$$\begin{aligned} H_0(K(x,M)) &= M/xM, \\ H_1(K(x,M)) &= \mathtt{ker}(m \mapsto xm), \\ H_i(K(x,M)) &= 0 \qquad \text{if } i > 1. \end{aligned}$$

So far so good, but what does this complex has to do with Tor? A lot in the case when A is noetherian.

Assume further that A is noetherian, then if x is not a zero divisor of A we have an isomorphism

$$H_i(x, M) \cong \mathtt{Tor}_i^A(A/x, M).$$

For some of the details see [**Ser**] §

We return to the proof (so $A = \mathcal{O}_P$ and is not obviously related to the A in the Kozsul complex). We have to compare

$$\mathtt{dim}_k A/(\mathfrak{p}_V + \mathfrak{p}_U) \qquad \text{and} \qquad \chi^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U).$$

By definition we have

$$\begin{aligned} \chi^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U) =& \mathtt{length}_A(A/\mathfrak{p}_V, A/\mathfrak{p}_U) - \mathtt{length}_A(\mathtt{Tor}_1^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U)) \\ &+ \mathtt{length}_A(\mathtt{Tor}_2^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U)). \end{aligned}$$

Now take in the Kozsul complex $A = A$, $x \in A$, such that $(x) = \mathfrak{p}_V$ and $M = A/\mathfrak{p}_U$. The assumptions from Bezout's Theorem (the curves do not share components) tell us that x (note that this is just the dehomogenization of F) is not a zero divisor of M and by our construction of the Kozsul complex we get:

$$\begin{aligned} A/\mathfrak{p}_V \otimes A/\mathfrak{p}_U &\cong H_0(x, A/\mathfrak{p}_U) = (A/\mathfrak{p}_U)/(\mathfrak{p}_V \cdot A/\mathfrak{p}_U), \\ \mathtt{Tor}_1^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U) &\cong H_1(x, A/\mathfrak{p}_U) = \mathtt{ker}(m \mapsto xm), \\ \mathtt{Tor}_2^A(A/\mathfrak{p}_V, A/\mathfrak{p}_U) &\cong H_2(x, A/\mathfrak{p}_U) = 0. \end{aligned}$$

Because the curves do not share components we see that $\mathtt{ker}(A/\mathfrak{p}_U \to x \cdot A/\mathfrak{p}_U) = 0$ and thus $\mathtt{Tor}_1^A$ is zero. Moreover

$$(A/\mathfrak{p}_U)/(\mathfrak{p}_V \cdot A/\mathfrak{p}_U) \cong A/(\mathfrak{p}_V + \mathfrak{p}_U).$$

It follows that we must show that

$$\mathtt{dim}_k A/(\mathfrak{p}_V + \mathfrak{p}_U) = \mathtt{length}_A(A/(\mathfrak{p}_V + \mathfrak{p}_U))$$

but this is clear because any A-submodule of $A/(\mathfrak{p}_V + \mathfrak{p}_U)$ is a k-subvector space and vice versa. This finishes the proof. □

An Example: The following example has been taken from [**Ful2**] Consider affine 4-space $\mathbb{A}^4$. Let

$$V = \mathcal{V}(x_1 - x_3, x_2 - x_4), \qquad U = \mathcal{V}(x_1x_3, x_1x_4, x_2x_3, x_2x_4).$$

Notice that $(x_1, x_2) \cap (x_3, x_4) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$ that is $U = U_1 \cup U_2$ where

$$U_1 = \mathcal{V}(x_1, x_2), \qquad U_2 = \mathcal{V}(x_3, x_4).$$

We compute the intersection multiplicity at $P = (0, 0, 0, 0)$ with the definition that we have for curves. That is we compute

$$\mathtt{dim}_k \left(\frac{k[x_1, x_2, x_3, x_4]_{(x_1,x_2,x_3,x_4)}}{\mathcal{I}(V) + \mathcal{I}(U)} \right)$$

Localization and quotients commute, so we can compute

$$\mathtt{dim}_k \left(\frac{k[x_1, x_2, x_3, x_4]}{(x_1 - x_3, x_2 - x_4) + (x_1x_3, x_1x_4, x_2x_3, x_2x_4)} \right)_{(\overline{x_1},\overline{x_2},\overline{x_3},\overline{x_4})}.$$

But

$$\left(\frac{k[x_1, x_2, x_3, x_4]}{(x_1 - x_3, x_2 - x_4) + (x_1x_3, x_1x_4, x_2x_3, x_2x_4)} \right)_{(\overline{x_1},\overline{x_2},\overline{x_3},\overline{x_4})} \cong \left(\frac{k[\overline{x_1}, \overline{x_2}]}{\overline{x_1}^2, \overline{x_1x_2}, \overline{x_2}^2} \right)_{(\overline{x_1},\overline{x_2})}$$

and this ring has dimension 3.

On the other hand, let us compute the intersection of U_1 with V and U_2 with V. For example,

$$\frac{k[x_1, x_2, x_3, x_4]}{(x_1 - x_3, x_2 - x_4) + (x_1, x_2)} \cong k \cong k_{(\overline{x_1})}$$

and after localizing at $\overline{x_1}$ this does not change, so the intersection multiplicity of $V \cap U_1$ at $P = (0, 0, 0, 0)$ is 1. Similarly, for $V \cap U_2$. And thus it seems that the intersection multiplicity of V and U at $(0, 0, 0, 0)$ is 2.

Clearly, $2 \neq 3$, so indeed we do need the `Tor`-formula that Serre gives.

5. References

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