

COMMUTATIVE ALGEBRA – PROBLEM SET 1

1. Prove that the maximal ideal of polynomials with constant term 0 in the ring $k[X_1, \dots, X_n]$ of polynomials in n variables (k is a field) cannot be generated by fewer than n elements. (This is true for any maximal ideal in $k[X_1, \dots, X_n]$ – it will be a consequence of the general theory later on.)

2. (See text, problem 16 on page 12.) For the rings $A = \mathbb{Z}, \mathbb{R}[X], \mathbb{Z}[X]$ list all the prime ideals in $\text{Spec}A$ (You don't have to prove that your lists are complete.) Describe the topology as well as you can.

Problems 3-10 are essentially the problems 17-20 (on page 12-13 in the text).

3. Let A be a ring. For $f \in A$ define $D(f) = \text{Spec}A \setminus V(f)$. Prove that the open sets $D(f)$ form a basis for the topology of $\text{Spec}A$.

4. Prove that the map $I \mapsto V(I)$ defines a natural bijection

$$\{I \subset A \text{ with } I = \sqrt{I}\} \longrightarrow \{T \subset \text{Spec}A \text{ closed}\}$$

A topological space X is called *quasi-compact* if for any open covering $X = \bigcup_{i \in I} U_i$ there is a finite subset $\{i_1, \dots, i_n\} \subset I$ such that $X = U_{i_1} \cup \dots \cup U_{i_n}$.

5. Prove that $\text{Spec}A$ is quasi-compact for any ring A .

A topological space X is said to verify the separation axiom T_0 if for any pair of points $x, y \in X$, $x \neq y$ then either there is an open subset of X containing x but not y , or there is an open subset of X containing y but not x . We say that X is *Hausdorff* if for any pair $x, y \in X$, $x \neq y$ there are disjoint open subsets U, V such that $x \in U$ and $y \in V$.

6. Show that $\text{Spec}A$ is not Hausdorff in general. Prove that $\text{Spec}A$ is T_0 . Give an example of a topological space X that is not T_0 .

A topological space X is called *irreducible* if X is not empty and if $X = Z_1 \cup Z_2$ with $Z_1, Z_2 \subset X$ closed, then either $Z_1 = X$ or $Z_2 = X$. A subset $T \subset X$ of a topological space is called irreducible if it is an irreducible topological space with the topology induced from X . The definitions imply that T is irreducible if and only if the closure \bar{T} of T in X is irreducible.

7. Prove that $\text{Spec}A$ is irreducible if and only if $\text{Nil}(A)$ is a prime ideal and that in this case it is the unique minimal prime ideal of A .

8. Prove that a closed subset $T \subset \text{Spec}A$ is irreducible if and only if it is of the form $T = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$.

A point x of an irreducible topological space X is called a *generic point* of X if X is equal to the closure of the subset $\{x\}$. In a T_0 space X every irreducible closed subset has at most one generic point.

9. Prove that in $\text{Spec}A$ every irreducible closed subset *does* have a generic point. In fact show that the map $\mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}}$ is a bijection of $\text{Spec}A$ with the set of irreducible closed subsets of X .

10. Give an example to show that an irreducible subset of $\text{Spec}\mathbb{Z}$ does not necessarily have a generic point.