COMMUTATIVE ALGEBRA – PROBLEM SET 2

A topological space X is called *Noetherian* if any decreasing sequence $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ of closed subsets of X stabilizes.

1. Show that if the ring A is Noetherian then the topological space Spec$A$ is Noetherian. Give an example to show that the converse is false. A maximal irreducible subset $T \subset X$ is called an *irreducible component* of the space X. Such an irreducible component of X is automatically a closed subset of X.

2. Prove that any irreducible subset of X is contained in an irreducible component of X.

3. Prove that a Noetherian topological space X has only finitely many irreducible components, say $X_1, \ldots, X_n$, and that $X = X_1 \cup X_2 \cup \ldots \cup X_n$. (Note that any X is always the union of its irreducible components, but that if $X = \mathbb{R}$ with its usual topology for instance then the irreducible components of X are the one point subsets.)

4. Show that irreducible components of Spec$A$ correspond to minimal primes of A.

A point $x \in X$ is called closed if $\overline{\{x\}} = \{x\}$.

5. Show that closed points of Spec$A$ correspond to maximal ideals of A.

6. Let I and J be ideals of A. What is the condition for $V(I)$ and $V(J)$ to be disjoint?

7. Let I and J be ideals of A. Show that if $I + J = A$ then $IJ = I \cap J$.

Show that the canonical injective morphism

$$A/(I \cap J) \to A/I \times A/J$$

is an isomorphism if and only if $I + J = A$.

A topological space X is called *connected* if it is not the union of two nonempty disjoint open subsets.

8. (This is essentially problem 22 in the text.) Show that Spec$A$ is disconnected iff $A \cong B \times C$ for certain nonzero rings B, C. More generally, let $A = \prod_{i=1}^{n} A_i$ be the direct product of rings A_i. Show that Spec$A$ is the disjoint union of open (and closed) subspaces X_i, where X_i is canonically homeomorphic with SpecA_i. Show that this fails when A is an infinite product of copies of $\mathbb{F}_2$ for example.

9. Compute Spec$k[x, y]$, where k is algebraically closed. [Hint: use the morphism

$$\phi : \mathrm{Spec}k[x, y] \to \mathrm{Spec}k[x]$$

if $\phi(\mathfrak{p}) = (0)$ then localize with respect to $S = \{f \in k[x] \mid f \neq 0\}$ and use result of lecture on localization and Spec.] (Why do you think this is called affine 2-space?)

10. Let M be a finitely generated A-module. Show that a surjective A-linear morphism $\phi : M \to M$ is an isomorphism. [Hint: Give M an $A[X]$-module structure and apply the Corollary of the Cayley-Hamilton theorem.]

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