

COMMUTATIVE ALGEBRA – PROBLEM SET 3

Let A be a ring and $S \subset A$ a multiplicative set.

1. Show that $M \mapsto S^{-1}M$ gives an exact functor from the category of A -modules to the category of $S^{-1}A$ -modules (i.e., it preserves exact sequences). Deduce that localization preserves kernels, cokernels, and intersections (if N, P are A -submodules of M then $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$).
2. Show that if M is an A -module, then $S^{-1}M \cong S^{-1}A \otimes_A M$.
3. Prove that if I is an ideal in A then $S^{-1}A/S^{-1}I \cong T^{-1}(A/I)$, where T is the image of S via the quotient map $\phi : A \rightarrow A/I$.
4. Find all the primes \mathfrak{p} and describe all the localizations $A_{\mathfrak{p}}$ for the ring $A = k[x, y]/(xy)$, where k is an algebraically closed field.
5. Find $\text{Spec}k[[x, y]]$.
6. Find an injection $M_1 \rightarrow M_2$ of A -modules such that $M_1 \otimes N \rightarrow M_2 \otimes N$ is not injective in the following cases:
 - a. $A = k[x, y]$ and $N = (x, y) \subset A$. (Here and below k is a field.)
 - b. $A = k[x, y]$ and $N = A/(x, y)$.
7. Let A be a ring and M an A -module. Let f_{α} be a collection of elements of A such that

$$\text{Spec}A = \bigcup D(f_{\alpha}).$$

- a. Show that if $M_{f_{\alpha}}$ is a finitely generated $A_{f_{\alpha}}$ -module, then M is a finitely generated A -module.
 - b. Show that if $M_{f_{\alpha}}$ is a flat $A_{f_{\alpha}}$ -module, then M is a flat A -module. (Finite generation and flatness are local properties- here we mean local in a different sense than we talked about in lecture!)
8. Let (I, \geq) be a partially ordered set which is directed. Let A be a ring and let $(N_{\alpha}, \varphi_{\alpha, \beta})$ be a directed system of A -modules indexed by I . Suppose that M is another A -module. Prove that

$$\lim_{\alpha \in I} M \otimes_A N_{\alpha} \cong M \otimes_A \left(\lim_{\alpha \in I} N_{\alpha} \right).$$

9. Prove that any module over any ring is the limit of its finitely generated submodules.
10. Problem 19 from page 46 in the text.
10. Problems 12 and 13 from page 45 in the text.