COMMUTATIVE ALGEBRA – PROBLEM SET 4

Let A be a ring and let M be an A-module.

1. Show that flatness is a local property, i.e., M is flat if and only if the localization $M_{\mathfrak{p}}$ is flat as an $A_{\mathfrak{p}}$-module, for any $\mathfrak{p}$ prime ideal (same for maximal ideals).

2. Problem 15, page 33, in the text.

3. Problem 21, pages 45-46, in the text.

4. Problem 20, page 46, in the text.

5. (Optional) – This is a generalization of problem 23, page 47, in the text.

Let $\mathfrak{p}$ be a prime ideal in A. Let $f, g \notin \mathfrak{p}$ be such that $D(g) \subseteq D(f)$.

i. Show that $g^n = f.h$ for some $h \in A$ and $n > 0$.

ii. Define a map

$$u_{f,g} : M_f \to M_g, \quad u_{f,g}(a/f^k) = ah^k/g^{nk}.$$

Show that the map $u_{f,g}$ is well-defined and that $(M_f, u_{f,g})$ is a direct system of A-modules (the index set is the set of $f \in A$, such that $f \notin \mathfrak{p}$, with the partial order given by inclusion of basic open sets $D(f)$).

iii. Show that the localization $M_{\mathfrak{p}}$ is the direct limit of the direct system $(M_f, u_{f,g})$.

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