COMMUTATIVE ALGEBRA – PROBLEM SET 6

1. Let $\mathfrak{p}$ be a prime ideal in $\mathbb{C}[X_1, \ldots, X_n]$ and let $Z \subset \mathbb{C}^N$ be the locus of those $a \in \mathbb{C}^N$ such that $f(a) = 0$ for any $f \in \mathfrak{p}$. For $n \geq 1$ let $\mathfrak{p}^{<n>}$ be the set of polynomials in $\mathbb{C}[X_1, \ldots, X_n]$ that vanish to order at least n along Z (i.e., all derivatives of order $\leq n-1$ vanish on Z). In this problem assume the Hilbert Nullstellensatz. Show that:

(a) $\mathfrak{p}^{<n>}$ contains $\mathfrak{p}^n$.
(b) $\mathfrak{p}^{<n>}$ is a $\mathfrak{p}$-primary ideal.
(c) $\mathfrak{p}^{<n>}$ contains the n-th symbolic power $\mathfrak{p}^{(n)}$ (the $\mathfrak{p}$-primary component of $\mathfrak{p}^n$).

(Hint for part (c): show in general that if J is the $\mathfrak{p}$-primary component of an ideal I, and $\mathfrak{p}$ is an isolated prime, then J is contained in any $\mathfrak{p}$-primary ideal that contains I.)

2. Assume that the zero ideal is decomposable in the ring A (for example if A is Noetherian). Show that the set of zero-divisors in A is the union of the associated primes of the zero ideal.

Let $\phi : A \to B$ be a homomorphism of rings.

We say that the *going-up theorem* holds for ϕ if the following condition is satisfied:

(GU) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $\mathfrak{q} \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$, there exists $\mathfrak{q}' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$ such that $\mathfrak{q} \subset \mathfrak{q}'$.

Similarly, we say that the *going-down theorem* holds for ϕ if the following condition is satisfied:

(GD) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $\mathfrak{q}' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$, there exists $\mathfrak{q} \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$ such that $\mathfrak{q} \subset \mathfrak{q}'$.

3. In each of the following cases determine whether (GU), (GD) holds, and explain why.

(a) k is a field, $A = k$, $B = k[x]$.
(b) k is a field, $A = k[x]$, $B = k[x, y]$.
(c) $A = \mathbb{Z}$, $B = \mathbb{Z}[1/11]$.
(d) k is an algebraically closed field, $A = k[x, y]$, $B = k[x, y, z]/(x^2 - y, z^2 - x)$.
(e) $A = \mathbb{Z}$, $B = \mathbb{Z}[i, 1/(2 + i)]$.
(f) $A = \mathbb{Z}$, $B = \mathbb{Z}[i, 1/(14 + 7i)]$.
(g) k is an algebraically closed field, $A = k[x]$, $B = k[x, y, 1/(xy - 1)]/(y^2 - y)$.

4. Let k be an algebraically closed field. Compute the image in $\mathrm{Spec}(k[x, y])$ of the following maps (geometric interpretation helps!):

(a) $\mathrm{Spec}(k[x, y, a, b]/(ax - by - 1)) \to \mathrm{Spec}(k[x, y])$.
(b) $\mathrm{Spec}(k[t, 1/(t - 1)]) \to \mathrm{Spec}(k[x, y])$, induced by $x \mapsto t^2$, and $y \mapsto t^3$.
(c) $k = \mathbb{C}$ (complex numbers), $\mathrm{Spec}(k[s, t]/(s^3 + t^3 - 1)) \to \mathrm{Spec}(k[x, y])$, where $x \mapsto s^2$, $y \mapsto t^2$.

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