COMMUTATIVE ALGEBRA – PROBLEM SET 8

1. Let A be a ring and $\ldots \subseteq I_2 \subseteq I_1 \subseteq A$ be a filtration of ideals. Let $\hat{A}$ be the completion with respect to the filtration $\{I_n\}$. Let $\hat{I_n}$ be the kernel of the canonical projection map $\hat{A} \to A/I_n$. Show that $\hat{A}$ is complete with respect to the filtration $\{\hat{I_n}\}$.

2. Let A be a ring and I, J two ideals such that there is a k, n such that $I^n \subseteq J$ and $J^k \subseteq I$. Show that the completions of A with respect to I and J are isomorphic.

3. Let A be a ring and $\mathfrak{m}$ a maximal ideal. Let $\hat{A}$ be the completion of A with respect to $\mathfrak{m}$. Show that $\hat{A}$ is a local ring. Let $\hat{A_{\mathfrak{m}}}$ be the completion of the localization $A_{\mathfrak{m}}$ of A at $\mathfrak{m}$. Show that $\hat{A_{\mathfrak{m}}} \cong \hat{A}$.

4. Problem 10 (b, c) p. 115 in the text.

5. Show that $f^n = u$ has a solution in the power series ring $k[[x_1, \ldots, x_k]]$ if u is an invertible series (assume k is algebraically closed and n does not divide the characteristic of k).

Some motivation for the word "completion":

Given a filtration of abelian groups $\ldots \subseteq G_2 \subseteq G_1 \subseteq G$, define the *Krull topology* on G by taking the subsets G_n as a base for the open neighborhoods of 0 and imposing that the addition is continuous (so the sets $x + G_n$ for $x \in G$, form a base for the open sets of the topology). Note, if G is a ring then the multiplication is continuous.

A *Cauchy sequence* with respect to the Krull topology is a sequence of elements $x_n \in G$ such that for each open neighborhood U of 0, there is a k such that for any $n, m > k$ we have $x_n - x_m \in U$. Two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ are *equivalent* if for each open neighborhood U of 0, there is a k such that for any $n > k$ we have $x_n - y_n \in U$.

6. Show that the set of Cauchy sequences forms a group under componentwise addition and that two sequences are equivalent if and only if their difference is equivalent 0.

7. Show that the set Cauchy sequences modulo the subgroup of sequences equivalent to 0 is isomorphic (as a group) to the completion $\hat{G}$ with respect to the filtration $\{G_n\}$. Generalize this for the case when G is a ring.

Remark: seen in this way the completion depends only on the topology, not on the choice of the filtration. (For example, you could try to prove problem 2 by showing that the "I-adic topology" is the same as the "J-adic topology".)

Date: April 2, 2008.