COMMUTATIVE ALGEBRA – PROBLEM SET 9

1. Find all the additive functions on the set of finitely generated A-modules when A is 1) a field; 2) $\mathbb{Z}$.

2. Let k be a field and let $S = k[x, y]$ be the graded ring with grading given by $\deg(x) = 2$ and $\deg(y) = 3$. Compute the Hilbert function $h_S(n)$ of S. Is there a Hilbert polynomial in this case? Same question for $S = k[x, y]/(x^2, xy)$.

Some more problems on completions:

3. Let $\phi : A \to B$ be a morphism of rings and assume that B is complete with respect to an ideal J. Let $f_1, \ldots, f_n \in J$. Show that there is a unique morphism of A-algebras $\psi : A[[x_1, \ldots, x_n]] \to B$ such that $\psi(x_i) = f_i$ for all i. Moreover, show that:

a. If $f_1, \ldots, f_n$ generate J and the induced map $\overline{\phi} : A \to B/J$ is surjective, then ψ is surjective. (You may use the fact that the induced map on associated graded rings

$$\mathrm{gr}(\psi) : A[x_1, \ldots, x_n] = \mathrm{gr}_{(x_1,\ldots,x_n)} A[[x_1, \ldots, x_n]] \to \mathrm{gr}_J B$$

is surjective.)

b. If $\mathrm{gr}(\psi)$ is injective then ψ is injective.

4. Prove (for $n = 1$ and 2) the inverse function theorem:

Let $f_1, \ldots, f_n$ be power series in the ideal $(x_1, \ldots, x_n)$ of $A[[x_1, \ldots, x_n]]$ and let ψ be the morphism:

$$\psi : A[[x_1, \ldots, x_n]] \to A[[x_1, \ldots, x_n]], \quad \psi(x_i) = f_i.$$

Let $J(x)$ be the Jacobian matrix (entries $(\partial f_i/\partial x_j)$.

Then ψ is an isomorphism if and only if $J(0)$ is a unit in A.

5. (Plane curve singularities - Optional) Let k be an algebraically closed field. In general, if A is a finitely generated k-algebra, we say that a point $\mathfrak{p} \in \mathrm{Spec}(A)$ is *non-singular* if $A_\mathfrak{p}$ is a *regular local ring* (a local ring $(A, \mathfrak{m}, k)$ such that $\dim A = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$). The completion of a regular local ring is also a regular local ring (proof later!). Cohen's structure theorem (a theorem which we did not get to prove) says that if $(A, \mathfrak{m}, k)$ is a Noetherian complete local ring which is regular and contains a field, then $A \cong k[[x_1, \ldots, x_n]]$ (where $n = \dim A$). It follows that when $\mathrm{Spec}A$ is a curve (i.e., $\dim A = 1$) a point $\mathfrak{p}$ is non-singular if and only if $\hat{A}_\mathfrak{p} \cong k[[t]]$.

Now let f be an irreducible polynomial in $k[x, y]$ be such that $f(0, 0) = 0$ Denote by $\overline{x}$, $\overline{y}$ the classes of x and y in the ring $k[x, y]/(f)$. Let $\mathcal{O}_p$ be the localization at $(\overline{x}, \overline{y})$ of the ring $k[x, y]/(f)$ and let $\hat{\mathcal{O}}_p$ be the completion of $\mathcal{O}_p$ with respect to its maximal ideal $(\overline{x}, \overline{y})$. Let

$$f = f_r + f_{r+1} + \ldots + \ldots f_d,$$

where f_e is a homogeneous polynomial in x, y of degree e (so f_r is the smallest degree term of f; by assumption $r \geq 1$). The number r is the *multiplicity* of f at p.

a. Show that $\hat{\mathcal{O}}_\mathfrak{p} \cong k[[t]]$ if and only if $r = 1$.

If $r \geq 2$ the singular point p is *an* r*-fold point.* We call this a double point if $r = 2$, a triple point if $r = 3$, etc. We say that p is *an ordinary* r*-fold point* if f_r is a product of distinct r linear factors.

b. Show that if p is an ordinary double point then there is a change of coordinates, i.e., an isomorphism

$$\psi : k[[x, y]] \to k[[u, v]]$$

such that $\psi(f) = v^2 - u^2$.

c. (Hard!) Show that if p is a double point then there is a change of coordinates ψ such that $\psi(f) = v^2 - u^{n+1}$ $(n \geq 1)$. This is called an A_n curve singularity. When $n = 1$, this is called a *node*; when $n = 3$, it is a *cusp*; when $n = 3$, it is a *tacnode*.

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