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1. Introduction

Smoothness is one of the most important properties studied in differential geometry, and so it stands to reason that it will also be important in algebraic geometry. It is not obvious, however, how the notion of smoothness should be generalized. In the case of varieties, one can make sense of the Jacobian criterion from differential geometry and it gives the correct notion; however, it is unclear how to determine something like the singular locus of $\text{Spec}(\mathbb{Z}[\sqrt{5}])$. We will primarily be interested in the regular local ring definition of smoothness, and we will explore it with a series of examples designed to build intuition.

2. The Jacobian Criterion

Let X be an affine variety over a field k. Then we know that X is isomorphic to the spectrum of a ring of the form $k[x_1, ..., x_n]/(f_1, .., f_k)$ which we can think of (roughly) as the intersection of the zero loci of the equations $f_1, ..., f_k$. In this case, we might guess, based on our intuition from differential geometry, that a k-rational point $\alpha = (\alpha_1, ..., \alpha_k)$ is nonsingular if and only if the rank of the Jacobian matrix whose $(i, j)^{th}$ entry is $\frac{\partial}{\partial x_i} f_j$ is the maximum possible when the entries are evaluated at α.

It turns out this "obvious" definition gives the correct answer, but has several apparent flaws. One is that it is not intrinsic to X, but rather to the representation of X. This is a minor issue, and it can be proven that it does not actually depend on the presentation of X. Another problem is that this only works for k-rational points, where as we would like a definition that at least worked for closed points. Also, it would be nice if the definition generalized to points of arbitrary schemes, rather than simply working for varieties. All of these issues can be fixed with the notion of a regular local ring.

3. Regular Local Rings

A local ring $(A, \mathfrak{m})$ is said to be regular if the dimension of A is equal to the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $A/\mathfrak{m}$. We define a point of the scheme $X = \text{Spec}(R)$ corresponding to an ideal $\wp$ of the ring R to be regular (what algebraic geometers call non-singular) if the ring $R_\wp$ is a regular local ring. A scheme which is regular at all of its points is said to be regular. The vector space $\mathfrak{m}/\mathfrak{m}^2$ is known as the Zariski cotangent space, and this condition essentially asserts that the cotangent space have the same dimension as the scheme, which agrees with geometric intuition. We recall a theorem which is of fundamental importance to what we will do below.

Theorem 1. *If $(A, \mathfrak{m})$ is a local ring, then the dimension of A is less than or equal to the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as an $A/\mathfrak{m}$ vector space.*

We now give several examples to illustrate the idea. We choose direct computation over theory here in order to make what is going on as transparent as possible.

Example 1: The scheme $X = \text{Spec}(k)$ is regular. The unique point of X corresponds to the ideal $\wp = (0)$, so we must check if $k_{(0)} \cong k$ is regular. It clearly has dimension 0, as does $(0)k_{(0)}/(0)^2k_{(0)}$.

Example 2: The point corresponding to the ideal $\mathfrak{m} = (x_1, ..., x_n)$ in the ring $R = k[x_1, ..., x_n]$ is regular in the scheme $X = Spec(R)$. The dimension of $R_\mathfrak{m}$ is at least n because we have the sequence of containments $(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq ... \subsetneq (x_1, ..., x_n)$. On the other hand, the dimension is at most n since $\mathfrak{m}$ is generated by n elements, and hence $dim(R_\mathfrak{m}) = n$. It is also clear that $\mathfrak{m}/\mathfrak{m}^2$ is generated by the elements $x_1, ..., x_n$, and hence the point is regular. Note this argument is sufficient to show that all k-rational points of X are regular.

Example 3: The point p corresponding to the ideal $\mathfrak{m} = (x^2+1, x-y)$ in the ring $R = \mathbb{R}[x, y]$ in the scheme Spec(R) is regular. Geometrically this point corresponds to both of the complex points $(i, -i)$ and $(-i, i)$ in affine 2 space over $\mathbb{C}$. Note that p is not $\mathbb{R}$-rational. Even so, $\mathfrak{m}$ is generated by 2 elements, and so, as in example 2, we can conclude that $dim(R_\mathfrak{m}) = dim_\mathbb{R}(\mathfrak{m}/\mathfrak{m}^2) = 2$, and so p is regular.

Example 4: The scheme $X = Spec(\mathbb{Z})$ is regular. It is easy to verify that all of the localizations $\mathbb{Z}_{(p)}$ are one dimensional with principal maximal ideals, and hence are regular. Also $\mathbb{Z}_{(0)}$ is a field, and hence trivially a regular local ring.

Example 5: The point corresponding to the ideal $\wp = (2, 1 + \sqrt{-5})$ in the ring $R = \mathbb{Z}[\sqrt{-5}]$ in the scheme $Spec(R)$ is regular. Note that the ring R is a Dedekind domain, and so is 1-dimensional. However, R has class number 2, and so has non principal ideals such as $\wp$. It is, perhaps, surprising then that this ideal corresponds to a regular point of $Spec(R)$. The reason is that although the idea $\wp$ is nonprincipal, the ideal $\wp R_\wp$ *is* principal and generated by $1+\sqrt{-5}$. To verify this, note that $2=(1+\sqrt{-5})\frac{1-\sqrt{-5}}{3}$ and $\frac{1-\sqrt{-5}}{3}\in R_\wp$ since $3\notin\wp$.

Example 6: The point p corresponding to the ideal $\wp=(x,y)$ in the ring $\mathbb{Z}[x,y]/(y^2-x^3-x)$ in $X=\text{Spec}(R)$ is regular. Note first that p is not a closed point, as the ideal $\wp$ is not maximal. Hence, the geometric meaning of the regularity of this point is less clear. It is possible to show, however, that every closed point that specializes p (i.e. the points corresponding to the maximal ideals containing $\wp$) is regular, and so the regularity of p in some sense represents the generic regularity of the closed points that lie on $V(\wp)$. In order to verify that p is regular, we note that the dimension of the ring $R_\wp$ is 1. Next note that $\wp^2=(x^2,xy,y^2)=(x^2,xy,x^3+x)=(x)$, and hence $\wp R_\wp/\wp^2R_\wp$ is one dimensional, generated by y. Thus we conclude p is regular.

Thinking geometrically, the scheme X is comes with a map to $X\to Spec(\mathbb{Z})$ whose fibers correspond to interpreting the equation $y^2=x^3+x$ over each finite field $\mathbb{F}_p$ and over $\mathbb{Q}$. The closed points of $V(x,y)$ correspond to the point $(0,0)$ of each fiber, i.e. over each field $\mathbb{F}_p$.

Example 7: The the closed points of $Spec(\mathbb{C}[x,y]/(y^2-x^3-x))$ are regular. All such points correspond to ideals of the form $\mathfrak{m}=(x-\alpha,y-\beta)$ since $\mathbb{C}$ is algebraically closed. Note that we have $\mathfrak{m}^2=(x^2-2x\alpha+\alpha^2,(x-\alpha)(y-\beta),y^2-2y\beta+\beta^2)$, and that we can replace y^2 with x^3+x giving $\mathfrak{m}^2=(x^2-2x\alpha+\alpha^2,(x-\alpha)(y-\beta),x^3+x-2y\beta+\beta^2)$. Adding $(x-2\alpha)(x^2-2x\alpha+\alpha^2)$ to the last generator, we find that, after doing some factoring,

$$\mathfrak{m}^2=(x^2-2x\alpha+\alpha^2,(x-\alpha)(y-\beta),(3\alpha^2+1)(x-\alpha)-2\beta(y-\beta)).$$

Note that $3\alpha^2+1$ and 2β cannot both be 0 if (α,β) lies on the curve $y^2=x^3+x$. Thus we have a linear relation between $x-\alpha$ and $y-\beta$ in $\mathfrak{m}/\mathfrak{m}^2$, and hence $\mathfrak{m}/\mathfrak{m}^2$ is 1 dimensional, and so the point corresponding to $\mathfrak{m}$ is regular.

Example 8: The point of $X=Spec(\mathbb{Q}[x,y,t]/(y^2-x^3-x-t))$ corresponding to the ideal $\mathfrak{m}=(x,y,t)$ is nonsingular. It is clear that the underlying ring is 2 dimensional. The space X can be thought of as a family of elliptic curves parameterized by t, and in thinking of X this way, the point corresponds to the point $(0,0)$ of the elliptic curve $y^2=x^3+x$ which we studied above. We compute $\mathfrak{m}^2=(x^2,y^2,t^2,xy,xt,yt)=(x^2,x^3+x+t,...)=(x^2,x+t,...)$. Thus there is a linear relation between x and t in $\mathfrak{m}/\mathfrak{m}^2$, and hence $dim_{\mathbb{Q}}(\mathfrak{m}/\mathfrak{m}^2) = 2$, and so the point is regular as claimed.

Now that we have given several examples of points which are regular, it is natural that we examine points that fail to be regular.

Example 1: The most obvious geometric example of something that should be nonregular is the intersection of two crossed lines. For example, we can consider the point corresponding to the ideal $\mathfrak{m} = (x, y)$ in $X = Spec(k[x, y]/(xy))$. Note that the ideal $\mathfrak{m}^2$ is generated by x^2 and y^2 and so does not contain a linear combination of x and y. Hence $\mathfrak{m}/\mathfrak{m}^2$ is 2-dimensional. It is easy to check that the only prime ideals of $(k[x, y]/(xy))_{\mathfrak{m}}$ are (x), (y), and (xy), and hence the dimension of $(k[x, y]/(xy))_{\mathfrak{m}}$ is 1. Thus we see that, as expected, the intersection point of two lines is singular.

Example 2: As another example, consider the point of $Spec(\mathbb{R}[x, y]/(y^2 - x^3 - x^2))$ corresponding to the ideal $\mathfrak{m} = (x, y)$. Geometrically, this point should be singular because the curve X crosses itself at this point. It is easy to compute that $\mathfrak{m}^2 = (x^2, xy, y^2) = (x^2, xy) = (x)(x, y) = (x)\mathfrak{m}$, and hence it is clear that $\mathfrak{m}^2$ does not contain a linear relation between x and y. Hence $dim_{\mathbb{R}}(\mathfrak{m}/\mathfrak{m}^2) = 2$. However, it is easy to verify that $(\mathbb{R}[x, y]/(y^2 - x^3 - x^2))_{\mathfrak{m}}$ is 1 dimensional, and hence the point is not regular.

Example 3: For a more arithmetic example, consider the point corresponding to the ideal $\mathfrak{m} = (2, 1 + \sqrt{-3})$ in $Spec(\mathbb{Z}[\sqrt{-3}])$. The ring $\mathbb{Z}[\sqrt{-3}]$ is 1 dimensional because $\mathbb{Z}$ is and the map $\mathbb{Z} \to \mathbb{Z}[\sqrt{-3}]$ is integral. However, we have that $\mathfrak{m}^2 = (4, 2(1+\sqrt{-3}), -2+2\sqrt{-3}) = (2)\mathfrak{m}$, and so $dim(\mathfrak{m}/\mathfrak{m}^2) = 2$, and hence the point is not regular. Note this would not have happened if we had instead used the integral closure of $\mathbb{Z}[\sqrt{-3}]$ (that is $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$).

4. Factoriality and Normality

We now restrict, for the entirety of this section, to the case that $X = Spec(R)$ for R a finitely generated k-algebra which is also a domain, and let us suppose that k is algebraically closed. This is enough to guarantee that we have the following lemma.

Lemma 2. *The singular locus of X is closed.*

We won't prove this result (see Liu for a complete proof). However, note that by the Jacobian criterion the set of closed points which are singular can be found by writing R in the form $k[x_1, ..., x_n]/(f_1, ..., f_r)$ and computing the points which are zeros of the order $n - dim(X)$ minors of the Jacobian matrix. Hence the set of singular closed points is closed in the topology on $MaxSpec(X)$. The slightly stronger result above requires a more careful understanding of what it means for a non closed point to be regular.

Example: The singular locus of the scheme $\mathbb{C}[x, y]/(y^2 - x^3 - x^2)$ consists of the single point $\wp = (x, y)$, and hence has codimension 1. To check this, we simply compute the Jacobian matrix $[2y \ \ -3x^2 - 2x]$. In order for the point (x, y) to be singular, this matrix would have to have rank 0, i.e. both entries would have to be 0. This can only happen when $y = 0$, which (by the equation for the curve) can only happen when x is 0, 1 or -1. Checking each of these by hand, we conclude the only singular point occurs when x and y are both 0, which gives the point $\wp$ as claimed.

Another result, which is especially useful in the study of curves, is the following.

Proposition 3. *If R is integrally closed (i.e. if X is normal), then the codimension of the singular locus of X is at least 2.*

This is particularly useful if X is a curve, in which case we have the following corollaries, which follows immediately from the previous proposition.

Corollary 4. *If X is a curve, then it is regular if and only if R is integrally closed.*

Corollary 5. *If X is a curve and R is factorial, then X is regular.*

It follows from this corollary that $R = \mathbb{C}[x, y]/(y^2 - x^3 - x^2)$ cannot be integrally closed. Indeed, it is easy to see that y/x satisfies the equation $T^2 - x - 1$ and so is integral over R but is not an element of R.

This suggests a method of "smoothing" out irreducible affine curves with singularities. Namely, replacing the ring R with its integral closure $\tilde{R}$, since, by the above corollary, it is guaranteed that $\tilde{X} = Spec(\tilde{R})$ is regular, and there is an obvious map $\tilde{X} \to X$. This process is known as normalization.

Though it is possible to normalize varieties of arbitrary dimension, using the process described above, it will not in general produce a non-singular variety. In order to resolve singularities in higher dimensional varieties it is necessary to use the more complex construction of blow ups, and the result is no longer canonical.