Relaxing the integral test: an "elementary" challenge for the advanced calculus student

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The subject of calculus has its share of technical assumptions, typically arising from powerful theorems in analysis. Are there good counterexamples that give insight into these subtleties and remain within an elementary toolbox? This article describes just such an illustrative example, whose analysis requires a number of techniques discussed in a second calculus course. The integral test is usually stated as the following.

Integral Test. *Suppose that $f(x)$ is a continuous, positive and decreasing function on the interval $[k, \infty)$ with $a_n = f(n)$. Then the following hold:*

(i) If $\int_k^\infty f(x)dx$ is convergent so is $\sum_{n=k}^\infty a_n$. *(ii)* If $\int_k^\infty f(x)dx$ is divergent so is $\sum_{n=k}^\infty a_n$.

We aim to show that, in the statement of the integral test, one cannot relax the condition that the function be decreasing. The intuition here is that oscillatory, wave-like functions can peak at the integers but have very little mass elsewhere, giving an integral that will not be comparable to our original sum. That intuition alone is enough to inspire a host of piecewise-linear counterexamples to a generalization of the integral test (see [**1**] for a typical example). Instead, we eschew the piecewise world and look for an elementary function that will do the job. What we propose is

$$\mathcal{CS}(x) = \frac{(\cos^2(\pi x))^x}{x} .$$

which is graphed in Figure 1. A simple calculation shows that

$$\sum_{n=1}^\infty \mathcal{CS}(n) = \sum_{n=1}^\infty \frac{(\cos^2(\pi n))^n}{n} = \sum_{n=1}^\infty \frac{1}{n} ,$$

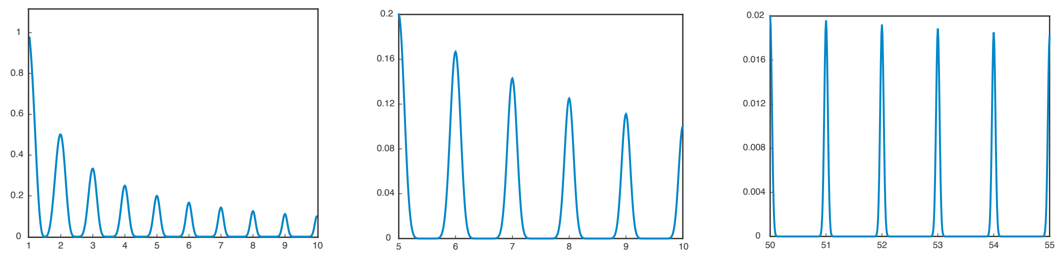
the harmonic series, which diverges. From Figure 1, we can make a guess as to why the integral has a chance of being finite.

Quite simply, all the mass of the function is centered at the integers! The integral, which measures area, will also capture how quickly the function decays between integer points, and hence will shrink quite rapidly. Still, we must prove that

$$\int_1^\infty \mathcal{CS}(x)dx < \infty .$$

The first step is to break up our integral into pieces.

$$\int_1^\infty \frac{(\cos^2(\pi x))^x}{x}dx = \sum_{n=1}^\infty \int_n^{n+1} \frac{(\cos^2(\pi x))^x}{x}dx \le \sum_{n=1}^\infty \int_n^{n+1} \frac{(\cos^2(\pi x))^n}{n}dx.$$



(a) Plot of $\mathcal{CS}(x)$ for $x \in [1, 10]$. (b) Plot of $\mathcal{CS}(x)$ for $x \in [5, 10]$. (c) Plot of $\mathcal{CS}(x)$ for $x \in [50, 55]$.

Figure 1.

An instructive, iterative calculation using repeated application of integration by parts shows that

$$\int_{n}^{n+1} \frac{(\cos^2(\pi x))^n}{n} dx = \frac{(2n-1)!!}{n(2n)!!} .$$

where the double factorial skips 2 each time, i.e. $8!! = 8 \times 6 \times 4 \times 2$. Now we have shown that our original integral is smaller than the following sum

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{n(2n)!!},$$

for which the ratio test proves inconclusive. However, if we define

$$A_n = \frac{(2n-1)!!}{(2n)!!} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \cdots \times \frac{3}{4} \times \frac{1}{2}$$

$$B_n = \frac{(2n)!!}{(2n+1)!!} = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} \times \cdots \times \frac{4}{5} \times \frac{2}{3},$$

a term-by-term comparison shows that each factor in A_n is smaller than the corresponding factor in B_n, hence $A_n < B_n$. So we have

$$A_n^2 \leq A_n B_n = \frac{(2n-1)!!}{(2n+1)!!} = \frac{1}{2n+1}.$$

Returning to our infinite sum, and employing the direct-comparison test,

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{n(2n)!!} = \sum_{n=1}^{\infty} \frac{A_n}{n} \leq \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2n+1}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} ,$$

which converges by the p-test, thus completing the proof.

References

1. Bernard R Gelbaum and John MH Olmsted. *Counterexamples in analysis*. Courier Corporation, 2003.

