Math 362: Lab 4

There is doubtless some truth to the old saw "a camel is a horse designed by a committee". However, in the first part of this lab we will explore the possibility that for certain decisions a committee may demonstrate better judgement than the individuals who make up the committee. Suppose that you are writing a new constitution and must make a decision about the size of a jury (N) and the rule by which it comes to a verdict (conviction takes place when k or more jurors vote to convict). In your country a typical citizen will vote to convict an innocent defendent 25% of the time. The same typical citizen will vote to convict a guilty defendent 80% of the time. Let $P(C|I)$ denote the probability that a *jury* convicts an innocent defendent. Let $P(C|G)$ be the probability that a jury votes to convict a guilty defendent.

Problem 1. What is the smallest jury size, N, so that there is a k for which,

$$P(C|I) \leq .001$$

and

$$P(C|G) \geq .99.$$

You want to make the mistake of convicting an innocent at most 1 time in 1000 and convict the guilty at a rate of 99%. Notice that making k larger makes the probability of conviction in either case smaller; this will help the first inequality to be true and make it harder for the second inequality to be true.

Now suppose the number of jurors is N and it takes k or more votes to convict. Let X_I be the number of jurors that vote to convict in the trial of an innocent. Then it is natural to model X_I as a binomial random variable of size N and probability $p = .25$. We have,

$$P(C|I) = \sum_{j=k}^{N} P(X_I = j).$$

The program 'R' makes it easy to calculate this binomial tail. It you enter,

$$pbinom(k, N, .25)$$

for some choice of $k \leq N$, R computes,

$$\sum_{j=0}^{k} \binom{N}{j} (.25)^j (.75)^{N-j},$$

which is $P(X_I \leq k)$. The probability we are interested in is a complementary probability,

$$P(X_I \geq k) = 1 - pbinom(k-1, N, .25).$$

Here is one way to start trying to answer the first question. Choose $N = 12$. Enter,

$$i = 1:12$$

$$x = 1:12$$

Then,

$$for \text{ (n in i) } x[n] = 1 - pbinom(n-1, 12, .25).$$

This produces a vector whose 7^{th} entry (for example) is the probability of a jury of 12 convicting an innocent provided 7 or more jurors must vote to convict. You can test if any entry is less than .001 by typing,

$$x <= .001$$

If you do this you will see a string of 12 answers TRUE or FALSE. The first TRUE answer occurs in the 9^{th} slot. Is $k = 9$ good enough to satisfy the other condition $P(C|G) \geq .99$? If X_G is the number of votes to convict a guilty defendent then X_G is naturally modeled as a binomial random variable with parameters N and $p = .8$. The probability that a jury with 12 jurors that requires 9 or more to convict will vote to convict a guilty defendent is,

$$1 - pbinom(8, 12, .8)$$

This is about .79 and this shows that 12 jurors will not do the trick. Hint: you do not need to consider $N > 25$.

Problem 2: Suppose Temujin and Ogadai are playing ping pong. In this version of the game the first to score 21 points wins. If Ogadai wins each point with probability .51 what is the probability that he wins the game? If the probability that Ogadai wins a point is .6 what is the probability that he wins the game? In both cases what is the probability that Ogadai wins the best 2 out of 3 games? Do you think that Ogadai should play in these circumstances?

It is useful to know the syntax for the negative binomial random variable in R. The quantity,

$$pnbinom(k, r, p)$$

is the probability that the first r successes (where individual success has probability p) occur in $k+r$ *or fewer trials* (it is a cumulative distribution). Thus $pnbinom(3, 2, .2)$ is the probability that the first 2 successes occur in either 2 or 3 or 4 or 5 independent trials with the probability of success on one trial equal to .2.