Comments on Homework 559 #2

4.4 (a) Fix a tableau associated with a partition $\lambda$ and write $a = a_\lambda$ and $b = b_\lambda$ for the associated row symmetrizer and column antisymmetrizer with $P = P_\lambda$ the subgroup that permutes the row elements among themselves and $Q = Q_\lambda$ the subgroup that permutes the column elements among themselves. Write $A = CG$ for the group algebra of the symmetric group $G = S_d$.

Consider the map $r_a$ defined by

$$V_\lambda = Aab \ni x \to xb \in Aba,$$

Note that $yab \to yaba \in Aaba \subset Aba$ so the image does end up in $Aba$. Now consider the map

$$Aba \ni x \to r_b x = xb \in Aab.$$

The composition $r_b r_a yab = y(ab)^2 = nyab$ from $Aab$ to $Aab$ is an isomorphism and this implies that $r_a$ is an isomorphism from $Aab$ to $Aba$. Since right multiplication commutes with the left group action it follows that $Aab$ and $Aba$ are isomorphic as $G$ modules. Note that for this argument to work smoothly you really want to regard $r_a$ as a map from $Aab$ to $Aba$ and not as a map into $Aaba$ (which it also is, of course).

(c) We want to show that $V_{\lambda} \otimes U'$ is isomorphic to $V_{\lambda}'$ where $\lambda'$ is the transpose of $\lambda$. First observe that since $(a_\lambda b_\lambda)^2 = n_\lambda a_\lambda b_\lambda$ it follows that $e_\lambda := \frac{a_\lambda b_\lambda}{n_\lambda}$ is a projection. Hence $\text{Tr}(e_\lambda) = \dim V_\lambda$ and we find that

$$n_\lambda = \frac{\text{Tr}(a_\lambda b_\lambda)}{\dim V_\lambda} = \frac{\dim A}{\dim V_\lambda} \tag{1}$$

since $\text{Tr}(g) = 0$ unless $g = e$ in the regular representation, in which case $\text{Tr}(e) = \dim A$. It is easy to use the same argument that worked to characterize $a_\lambda b_\lambda$ as the unique element (up to a multiple) that is invariant under left multiplication by all $p \in P_\lambda$ and right multiplication by $\text{sgn}(q)q$ for $q \in Q_\lambda$, to see that $b_\lambda a_\lambda$ is characterized as the unique element (again up to a multiple) that is invariant under right multiplication by $p \in P_\lambda$ and left multiplication by $\text{sgn}(q)q$ for $q \in Q_\lambda$. It follows that for some $m_\lambda$ we have $(b_\lambda a_\lambda)^2 = m_\lambda b_\lambda a_\lambda$. But then $f_\lambda := \frac{b_\lambda a_\lambda}{m_\lambda}$ is a projection on $A_{\lambda} a_\lambda$. Thus $\text{Tr}(f_\lambda) = \dim(A_{\lambda} a_\lambda)$. Part (a) shows that $\dim(A_{\lambda} a_\lambda) = \dim V_\lambda$ so that

$$m_\lambda = \frac{\text{Tr}(b_\lambda a_\lambda)}{\dim V_\lambda} = \frac{\dim A}{\dim V_\lambda}. \tag{2}$$

It follows from (1) and (2) that $n_\lambda = m_\lambda$.

Now we consider the characters of the representations $V_\lambda \otimes U'$ and $V_{\lambda}'$. Suppose that $g \in S_d$, then the character of $V_{\lambda}' \otimes U'$ is the product of the characters of $V_\lambda$ and $U'$ or

$$\text{sgn}(g) \chi_\lambda(g).$$

The character formula that appears just before equation (6) in the notes A Character Formula for Weyl Modules is,

$$\chi_\lambda(g) = \sum_{s \in G} e_\lambda(s^{-1} g^{-1} s) \tag{3}$$

Note that the character of $V_\lambda$ is not given by $\text{Tr}(g e_\lambda)$ since this is the trace of left multiplication by $e_\lambda$ followed by left multiplication by $g$ and we need instead right multiplication by $e_\lambda$. Furthermore since

$$e_\lambda = \frac{1}{n_\lambda} \sum_{p \in P, q \in Q} \text{sgn}(q)pq,$$

we find that,

$$e_\lambda(r) = \frac{1}{n_\lambda} \sum_{p \in P, q \in Q} \text{sgn}(q)\delta_{pq}(r),$$

1
where $\delta_{pq}(r) = 0$ unless $r = pq$ in which case $\delta_{pq}(pq) = 1$. Thus the trace of the representation $V_\lambda \otimes U'$ is given by

$$\frac{1}{n_\lambda} \sum_{s \in G} \sum_{p \in P, q \in Q} \text{sgn}(pq) \delta_{pq}(s^{-1} g^{-1} s) \quad (4)$$

Now let us consider the character of $V_\lambda$. According to the first part of this problem we can compute this character by finding the character of the representation $M_\lambda \otimes a_\lambda$. We know that $f_{\lambda'} = \frac{1}{n_{\lambda'}} b_\lambda a_{\lambda'}$ is the right projector for this representation so the formula for the character of $V_{\lambda'}$ is,

$$\chi_{\lambda'}(g) = \sum_{s \in G} f_{\lambda'}(s^{-1} g^{-1} s).$$

But

$$f_{\lambda'} = \frac{1}{n_{\lambda'}} \sum_{p \in P, q \in Q} \text{sgn}(p) pq,$$  \quad (5)

so

$$\chi_{\lambda'}(g) = \frac{1}{n_{\lambda'}} \sum_{s \in G} \sum_{p \in P, q \in Q} \text{sgn}(p) \delta_{pq}(s^{-1} g^{-1} s) \quad (6)$$

The only terms that contribute to this last sum are those for which $pq = s^{-1} g^{-1} s$. Thus in the sum $\text{sgn}(pq) = \text{sgn} g$ or $\text{sgn}(p) = \text{sgn}(gq)$. Substituting this in (6) we find that it agrees with the character (4) provided that $n_\lambda = n_{\lambda'}$. This is all we need to show to finish the proof. The normalization constant $n_{\lambda'}$ is the coefficient of $e$ in the expansion of

$$\left( \sum_{p, q} \text{sgn}(p) pq \right)^2.$$ 

This in turn is

$$n_{\lambda'} = \sum_{p, q, p', q'} \text{sgn}(p) \text{sgn}(p')$$ \quad (7)

where the sum is over all $p, p' \in P$ and $q, q' \in Q$ with $pq'q' = e$. But for this restriction we have $\text{sgn}(p) \text{sgn}(p') = \text{sgn}(q) \text{sgn}(q')$ and making this substitution in (7) we find that the sum becomes the normalization sum for $c_\lambda$ (i.e., $n_\lambda$). At this point one realizes that there was no need to show that $n_\lambda = m_\lambda$ as was done above (still, it is interesting).

4.13 The Frobenius formula for the dimension of $V_\lambda$ is

$$\dim V_\lambda = \frac{d!}{l_1! \cdots l_k!} \prod_{i < j} (d_i - d_j).$$

The problem is to use this formula to prove that

$$\dim V_\lambda = \frac{d!}{\prod \text{(hook lengths)}}.$$

Clearly it is enough to show that,

$$\prod \text{(hook lengths)} = \frac{l_1! \cdots l_k!}{\prod_{i < j} (l_i - l_j)},$$ \quad (8)

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ be a partition. Then $l_j = \lambda_j + k - j$. It is useful to note that the hook lengths for the elements in the first column of the Young diagram associated with the partition $\lambda$ are $l_1, l_2, \ldots, l_k$ reading from top to bottom. Now suppose that we start with the Young diagram $\lambda$ and make it bigger in one of two ways. (1) we add another column on the left without increasing the number of rows or (2) we add additional elements only to the first column. Every Young diagram can be constructed by a combination.
of such alterations applied to the trivial diagram with just 1 box so we may proceed inductively. First suppose that the hook length formula (8) is true for $\lambda$ and $\mu$ is obtained from $\lambda$ by adding another column without changing the number of rows. The hook lengths in the first column of $\mu$ are easily seen to be $l_1 + 1, l_2 + 1, \ldots, l_k + 1$ where $l_j$ are the hook lengths in the first column of $\lambda$. To go from the hook length formula (8) for $\lambda$ to the hook length formula for $\mu$ one needs only multiply (8) by the product,

$$
\prod_{j=1}^{k} (l_j + 1).
$$

The denominator on the right hand side of the hook length formula for $\mu$ is the same as the formula for $\lambda$ since all the $l_j$ have been translated by the same amount, 1. This is the inductive step for case (1).

Now suppose that $\mu$ is obtained from $\lambda$ by adding $n$ new elements in the first column. The hook lengths in the first column of $\mu$ are now given by,

$$
l_1 + n, l_2 + n, \ldots, l_k + n, n, n - 1, \ldots, 1
$$

Writing $l_j'$ for $j = 1, \ldots, k + n$ for this sequence one easily computes,

$$
\prod_{i < j \leq k + n} (l_i' - l_j') = \prod_{i < j \leq k} (l_i - l_j) \prod_{j=1}^{k} l_j (l_j + 1) \cdots (l_j + n - 1) \prod_{j=1}^{n-1} j!
$$

(9)

The product of the hook lengths for $\mu$ is obtained from the product of the hook lengths for $\lambda$ by first dividing by $\prod_{j=1}^{k} l_j$ and then multiplying by $\prod_{j=1}^{k+n} l_j'$. Doing this to the right hand side of (8) one finds,

$$
\frac{\prod_{j=1}^{k+n} l_j' \prod_{j=1}^{k} (l_j - 1)!}{\prod_{i < j \leq k+n} (l_i' - l_j')}.
$$

Multiplying the numerator and denominator of this last expression by the missing factors in (9) one finds,

$$
\frac{\prod_{j=1}^{k} l_j (l_j + n)! \prod_{j=1}^{n-1} j!}{\prod_{i < j \leq k+n} (l_i' - l_j')},
$$

This finishes the inductive proof (it is never wise to think too much about the $n = 1$ case).

4.19. If $V$ is the standard representation of $S_d$ show that

$$
\text{Sym}^2 V \simeq U \oplus V \oplus V_{(d, 2, 2)}
$$

and

$$
\text{Alt}^2 V \simeq V_{(d-2, 1, 1)}.
$$

This is a character calculation. We know that if $c(i)$ is the conjugacy class associated with the sequence $i = (i_1, i_2, \ldots, i_d)$ with $i_j = \# \text{ of cycles of size } j$, then $\chi_V(c(i)) = i_1 - 1$ which follows either from the Frobenius formula or more simply from the fact that $V$ is the complement of the identity representation inside a permutation representation and $i_1$ just counts the number of fixed points in the permutation representation. We also see that $\chi_V(c(i)^2) = i_1 + 2i_2 - 1$ by counting the number of fixed points for $c(i)^2$. Thus

$$
\chi_{\text{Sym}^2 V}(c(i)) = \frac{1}{2} \left( \chi_V(c(i))^2 + \chi_V(c(i)^2) \right) = \frac{1}{2} \left( (i_1 - 1)^2 + i_1 + 2i_2 - 1 \right),
$$

which is easily seen to be the same as,

$$
\chi_U(c(i)) + \chi_V(c(i)) + \chi_{(d-2, 2)}(c(i)),$$

using the formula in 4.15 for the last character. In a similar fashion the character for $\text{Alt}^2(V)$ is seen to be,

$$
\frac{1}{2} \left( (i_1 - 1)^2 - i_1 - 2i_2 + 1 \right),
$$

which is the character for $V_{(d-2, 1, 1)}$ according to 4.15.