Final Exam Math 559

December 3, 2006

1. Use the fact that $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a the simply connected covering for $n \geq 3$ to find a closed path in $\mathrm{SO}(n)$ which generates $\pi_1(\mathrm{SO}(n), e)$.

2. Use covering space theory to find all the continuous group homomorphisms from S^1 to S^1. Find the group of continuous automorphisms of the group S^1. Solve the same problem for the torus $\mathbf{T}^2 = S^1 \times S^1$. You can use (without proof) that any continuous additive map from $\mathbf{R}^n$ to $\mathbf{R}^n$ is real linear.

3. Find the translation invariant measure on S^1. Use Shur's lemma to determine all the continuous finite dimensional irreducible representations of S^1. Use these facts to identify the Peter-Weyl decomposition of $L^2(S^1)$ with Fourier series. Do the same for $\mathbf{T}^2 = S^1 \times S^1$.

4. The group $\mathrm{SO}(3)$ acts in an obvious way on the two sphere S^2. Show that there exists a volume form ω on S^2 which is invariant under the action of $\mathrm{SO}(3)$ and that this volume form is unique up to a constant multiplier. Show that $\mathrm{SO}(3)$ acts by unitary transformations on $L^2(S^2, \omega)$, the L^2 space of measurable functions, f, on S^2 such that:

$$\int_{S^2} |f|^2 \, \omega < \infty.$$

Note: The Peter-Weyl theorem implies that this space decomposes into finite dimensional irreducible pieces. An explicit decomposition is given by *spherical harmonics.*

5. Consider the map:

$$S^1 \times SU(n) \ni (z, U) \to zU \in U(n) \text{ where } z = e^{i\theta}.$$

Is this a homomorphism of groups? Is this a covering space map? What is $\pi_1(U(n), e)$?