Problem Set 4: Math 559

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1. Suppose that V is a finite dimensional complex vector space with a non degenerate symmetric bilinear form, $S(x,y)$. The form S is non-degenerate if the linear functional $x \to S(x,y)$ is non zero for every choice of $y \in V$. It is always possible to choose an orthonormal basis $\{e_j\}$for V so that for vectors,

$$x = \sum_j x_j e_j, \;\; y = \sum_j y_j e_j,$$

we have,

$$S(x,y) = \sum_j x_j y_j.$$

You don't need to prove this but the modifications needed for Gram-Schmidt are not hard. The (algebraic) tensor algebra over V is the direct sum,

$$TV := \bigoplus_{k=0}^{\infty} T^k V,$$

where $T^n V$ is the n fold tensor product of V with itself, $T^0 V = \mathbf{C}$ and the elements in the direct sum are non-vanishing in only finitely many summands. Let I denote the two sided ideal generated by elements of the form $x \otimes y + y \otimes x - S(x,y)1$, for $x, y \in V$. Define the Clifford Algebra, $\mathrm{Cliff}(V,S)$ to be the quotient,

$$TV/I.$$

Show that $e_{j_1,j_2\ldots,j_k} := e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k} + I$ for $j_1 < j_2 < \cdots < j_k$ is a basis for $\mathrm{Cliff}(V,S)$. Show that any representation of the Clifford relations,

$$\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{jk}, \text{ for } j,k = 1,2,\ldots \dim(V) \tag{1}$$

on a complex vector space W leads to a representation of $\mathrm{Cliff}(V,S)$ on W and vice versa (this is a basis dependent construction). Conclude from the representation theory for the Clifford group, $CL(n)$, that $\mathrm{Cliff}(V,S)$ has a unique irreducible representation class of dimension $2^{\frac{\dim V}{2}}$ when V is even dimensional and that $\mathrm{Cliff}(V,S)$ has two inequivalent irreducible representations of dimension $2^{\frac{\dim V - 1}{2}}$ when V is odd dimensional. Note that arbitrary representations of 1 will not lead to *unitary* representations of the associated finite group (we do not even suppose that that the representation space has an inner product). This means that the intertwining map for equivalent irreducible representations is non-singular but not necessarily unitary. One can restore the connection with unitary representations of the finite Clifford group by considering vector spaces V with a distinguished conjugation $x \to \bar{x}$ so that $\langle x, y\rangle := S(\bar{x}, y)$ is an inner product. The complex conjugation $x \to \bar{x}$ extends to uniquely to an anti-linear, anti-automorphism * of $\mathrm{Cliff}(V,S)$. A representation ρ of $\mathrm{Cliff}(V,S)$ on an inner product space W is then said to be a * representation iff $\rho(A^*) = \rho(A)^*$ where the * on the right is the usual Hermitian adjoint with respect to the inner product on W. The correspondence between unitary representations of the Clifford group generated by $\{e_1, e_2, \ldots, e_n\}$and * representations of $\mathrm{Cliff}(V,S)$ is restored provided that e_j is a real basis. That is $\bar{e_j} = e_j$. This is equivalent to $\gamma_j^* = \gamma_j$. For some purposes the extra structure (see the next problem) associated with a * involution is not helpful and it is useful to enlarge the representations of the Clifford algebra to include non * representations.

2. Let $O(V,S)$ denote the group of complex linear transformations, $r : V \to V$ such that,

$$S(rx, ry) = S(x, y).$$

Define,

$$\phi_r = 1 \oplus r \oplus r \otimes r \oplus \cdots \oplus \otimes^n r \oplus \cdots,$$

Show that when $r \in O(V,S)$ the transformation ϕ_r on TV induces an algebra automorphism,

$$\phi_r : \mathrm{Cliff}(V,S) \to \mathrm{Cliff}(V,S),$$

Suppose that V is even dimensional and $\rho : \mathrm{Cliff}(V,S) \to \mathrm{Hom}(W)$ is irreducible. Show that $\rho \circ \phi_r$ is also irreducible and hence that there exists $\Phi_r : W \to W$ so that,

$$\rho \circ \phi_r = \Phi_r \rho \Phi_r^{-1}.$$

Now let $\mathcal{G}$ denote the group of invertible linear transformations $g \in \mathrm{Hom}(W)$ such that there exists a linear map τ_g so that for all $v \in V$ we have,

$$g\rho(v)g^{-1} = \rho(\tau_g v).$$

Show that there is a short exact sequence of groups,

$$\mathbf{C} \to \mathcal{G} \to O(V,S) \to 0,$$

where the homomorphism from $\mathcal{G}$ onto $O(V,S)$ is given by τ. The group $\mathcal{G}$ is also sometimes called the Clifford group (it is finite dimensional but certainly not finite). A reduction of this exact sequence over the real special orthogonals to an extension by $\mathbf{Z}_2$ is possible and gives,

$$\mathbf{Z}_2 \to \mathrm{Spin}(n) \to SO(n) \to 0,$$

which turns out to be a model for the simply connected covering, Spin(n), of $SO(n)$. We will take this up more seriously later in the course. For now note that we can also express the fact that $g \in \mathcal{G}$ in the following way (note $\gamma_j = \rho(e_j)$),

$$g\gamma_j g^{-1} = \sum_k \tau_g^{kj} \gamma_k,$$

where τ_g^{kj} is just the matrix of τ_g with respect to the basis e_j. In the Quantum Physics, Clifford Algebras arise in the representation theory of the canonical anti-commutation relations which are important for discussing particles with anti-symmetric statistics. In that literature the group of transformations that linearly transform the generators of the Clifford algebra is called the group of Bogoliubov transformations ($=\mathcal{G}$). The larger group $\mathcal{G}$ (compared to Spin(n)) and infinite dimensional generalizations are important for applications to Physics.

Now suppose that V is odd dimensional. There are two irreducible representations, ρ_1 and ρ_2 of the Clifford algebra in this case so we can't argue that $\rho_1 \circ \phi_r$ is equivalent to ρ_1; it might be equivalent to ρ_2. To sort this out first prove that if $r \in O(V,S)$ then $\det r = \pm 1$. Next observe that the two representations ρ_1 and ρ_2 are distinguished by the action of the central element $\gamma := \gamma_1 \cdots \gamma_n$ (remember n is odd). Show that,

$$\phi_r \gamma = \det(r)\gamma.$$

Argue that $\phi_r \gamma = \det(r)\gamma + \text{lowerorderterms}$. Then argue that the element $\phi_r \gamma$ must still be central and so the lower order terms must vanish. Use this to show that if $\det r = 1$ then $\rho_j \circ \phi_r \simeq \rho_j$, that is, the automorphism ϕ_r does not change the equivalence class of the representation ρ_j. Thus for each element $r \in SO(V,S)$ (the orthogonals with determinant 1) there exist linear maps $\Phi_{r,j}$ acting on the representation space of ρ_j so that,

$$\rho_j \circ \phi_r = \Phi_{r,j} \rho_j \Phi_{r,j}^{-1}.$$