Problem set 8: Math 559

March 1, 2007

1. Show that a compact Lie group, G, has a Riemannian metric which is invariant under both right and left translation. (hint: start with an Ad invariant inner product at the identity). Show that one obtains a bi-invariant metric d on by,

$$d(p,q) = \inf_{\psi} \int_a^b \sqrt{\langle \psi'(s), \psi'(s) \rangle}\, ds,$$

where the infimum is over all piecewise smooth curves joining p to q.

2. In Simon's book there is an argument about the structure of the Weyl group that proceeds by constructing a path that joins an interior point in one Weyl chamber to an interior point in another by moving in the complement of union of the intersections $\tilde{\pi}_\alpha \cap \tilde{\pi}_\beta$ for $\alpha \neq \beta$. Show that this complement is path connected and that one can choose a path connecting interior points in different Weyl chambers that crosses the faces of the intermediate Weyl chambers only finitely many times.

3. Suppose that G is a Lie group. Show that,

$$\mathrm{Ad}(e^{tX}) = e^{t\mathrm{ad}(X)} = \sum_{k=0}^{\infty} \frac{t^n}{n!} \mathrm{ad}^n(X),$$

where $X \in LG$ and the exponential on the right is the usual power series for linear operators.

4. Determine the Weyl group for SO(2n) and also for SO(2n+1).