Linear and Weakly Nonlinear Control Systems

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Contents

Abstract

This paper presents the linear operator, Laplace Transform, its properties, and its application within the subject of Linear Control System Theory, which deals with the behavior of dynamical systems. Within Linear Control System Theory, we cover the three main system properties: stability, observability, and controllability, as well as, ways to design a control system. In the end, we utilize Laplace Transform along with Linear Control System Theory to solve the equations of motion for the gyropendulum and design a control system to augment its behavior.

1 Laplace Transform

1.1 Definition of Laplace Transform

The Laplace transform is a linear operator on a function, denoted by $\mathcal{L}\{f(t)\}$, and is the most commonly used integral transform. It can be used to solve linear differential and integral equations, but it is mainly used to solve ordinary differential equations by converting them to solvable algebraic equations. It transforms a function from the time-domain to the frequency-domain, where a function that is formerly a function of time is now a function of complex angular frequency. The Laplace Transform can be thought of as a generalized form of the continuous Fourier Transform. The relation between the Laplace transform and the Fourier transform can be clearly shown by finding the Fourier transform of $f(x)H(x)e^{-\gamma x}$.
The Fourier Transform is defined as

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dk \int_{-\infty}^{\infty} g(t)e^{ikt}dt. \tag{1.1}$$

Let $g(x) = f(x)H(x)e^{-\gamma x}$, where γ is a positive real number, where $H(x)$ is the *Heaviside step function*

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Then,

$$f(x)H(x)e^{-\gamma x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dk \int_{-\infty}^{\infty} f(t)H(t)e^{-\gamma t}e^{ikt}dt \tag{1.2}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dk \int_{0}^{\infty} f(t)e^{-\gamma t}e^{ikt}dt \tag{1.3}$$

Multiply both sides by $e^{\gamma x}$ to get,

$$f(x)H(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{x(\gamma-ik)}dk \int_{0}^{\infty} f(t)e^{-t(\gamma-ik)}dt \tag{1.4}$$

Let $s = \gamma - ik$, where s is a complex variable, we get

$$f(x)H(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{xs}ds \int_{0}^{\infty} f(t)e^{-ts}dt \tag{1.5}$$

which is the Laplace transform of f(t), where $f(t)$ is a real- or complex-valued function defined for all $t > 0$.

Definition 1.1. *Let f(t) be a function, real- or complex-valued, of time, t where* $t \in (0, \infty]$*, and s is a complex variable. The* **Laplace transform** *of f is defined as*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \lim_{R\to\infty} \int_0^R f(t)e^{-st}dt, \tag{1.6}$$

if the limit exists. The **inverse Laplace transform** *of F is defined as*

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds = \lim_{R\to\infty}\left[\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} F(s)e^{st}ds\right], \tag{1.7}$$

if the limit exists.

Finding the inverse Laplace transform of a function requires the use of contour integration techniques, such as the *Cauchy's Residue Theorem.*

Theorem 1.2 (Cauchy's Residue Theorem)**.** *Let f(z) be analytic inside and on a simple closed contour C, except for a finite number of isolated singular points* s_1*,* s_2*, ...,* s_n *located inside* C*. Then*

$$\oint f(z)dz = 2\pi i \sum_{i=1}^{N} a_i \tag{1.8}$$

where a_i *is the* ***residue*** *of* $f(z)$ *at* $z = s_i$*, denoted by* $a_i = Res\,(f(z); s_i)$*.*

Define $f(z)$ as

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where $\phi(z)$ is analytic in the neighborhood of $z = z_0$ and m is a positive integer. If $\phi(z_0) \neq 0, f(z)$ has a pole of order m. Therefore, the residue of $f(z)$ at z_0 can be calculated as

$$C_{-1} = \frac{1}{(m-1)!}\left(\frac{d^{m-1}}{dz^{m-1}}\phi\right)(z = z_0)\,. \tag{1.9}$$

Example 1.3. $f(t) = t^2 + 3t$

$$F(s) = \int_0^\infty \left(t^2 + 3t\right)e^{-st}dt = \int_0^\infty t^2e^{-st}dt + \int_0^\infty 3te^{-st}dt$$

$$\begin{aligned}\int_0^\infty t^2e^{-st}dt &= \left.\frac{-t^2e^{-st}}{s}\right|_0^\infty + \frac{2}{s}\int_0^\infty te^{-st}dt = \frac{2}{s}\int_0^\infty te^{-st}dt = \frac{2}{s}\left[\left.\frac{te^{-st}}{-s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}dt\right] \\ &= \frac{2}{s^2}\int_0^\infty e^{-st}dt = \frac{2}{s^2}\left[\left.\frac{-e^{-st}}{s}\right|_0^\infty\right] = \frac{2}{s^3}\end{aligned}$$

$$\int_0^\infty 3te^{-st}dt = 3\left[\left.\frac{te^{-st}}{-s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}dt\right] = \frac{3}{s}\int_0^\infty e^{-st}dt = \frac{3}{s}\left[\left.\frac{e^{-st}}{-s}\right|_0^\infty\right] = \frac{3}{s^2}$$

$$F(s) = \frac{2}{s^3} + \frac{3}{s^2}$$

Finding the inverse Laplace Transform of $F(s)$ necessitates the use of contour integration.

We have that

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-ik}^{\gamma+ik} F(s)e^{st}ds = \frac{1}{2\pi i}\oint_{C_R} F(z)e^{zt}dz = \frac{1}{2\pi i}\oint_{C_R}\left(\frac{2}{z^3}+\frac{3}{z^2}\right)e^{zt}dz = \frac{1}{2\pi i}\oint_{C_R}\frac{(2+3z)}{z^3}e^{zt}dz,$$

where C_R is the integral in the figure 1.

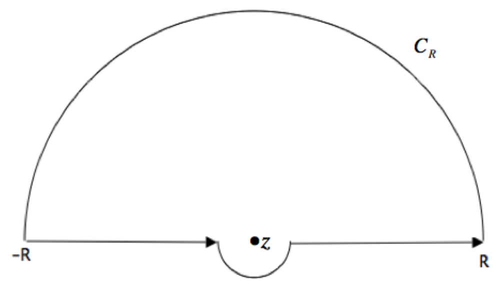


Figure 1: Contour C_R

Using equation 1.9, $\phi(z) = (2+3z)e^{zt}$, $z_0 = 0$, and $m = 3$, so,

$$C_{-1} = \frac{1}{2}\left.\frac{d^2}{dz^2}\phi(z)\right|_{z=0} = \frac{1}{2}\left(3zt^2 + 2t^2 + 6t\right)e^{zt}\Big|_{z=0} = t^2 + 3t,$$

and

$$f(t) = \frac{1}{2\pi i}\oint_{C_R} F(z)e^{zt}dz = \frac{1}{2\pi i}(2\pi i C_{-1}) = t^2 + 3t.$$

As a result, we have our original equation: $f(t) = t^2 + 3t$.

1.2 Properties of the Laplace Transform

For the following properties, α and β are constants and the functions, $f(t)$ and $g(t)$ arbitrary functions.

Linearity

$$\mathcal{L}\{\alpha f(t)\} = \alpha F(s) \tag{1.10}$$

Superposition

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s) \tag{1.11}$$

Frequency Shift

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a). \tag{1.12}$$

Time Shift

$$\mathcal{L}\left\{f(t-a)H(t-a)\right\} = e^{-sa}F(s). \tag{1.13}$$

Scaling

$$\mathcal{L}\left\{f(\alpha t)\right\} = \frac{1}{\alpha}F\left(\frac{s}{\alpha}\right). \tag{1.14}$$

1.3 Laplace Transform and Convolution

The convolution is defined as

$$h(t) = f * g = \int_0^t f(t-\xi)g(\xi)d\xi. \tag{1.15}$$

With the convolution, we can derive the equation

$$\mathcal{L}\left[h(t)\right] = \mathcal{L}\left[f(t)\right]\mathcal{L}\left[g(t)\right]. \tag{1.16}$$

We begin by finding the Laplace transform of $h(t)$ as defined in equation 1.15,

$$\begin{aligned}\mathcal{L}\left[h(t)\right] &= \int_0^\infty e^{-st}\left(\int_0^t f\left(t-\xi\right)g\left(\xi\right)d\xi\right)dt = \int_0^\infty \int_0^t e^{-st} f\left(t-\xi\right)g\left(\xi\right)d\xi dt \\ &= \int_0^\infty \int_0^t e^{-st} f\left(t-\xi\right)g\left(\xi\right)d\xi dt = \int_0^\infty \int_\xi^\infty e^{-st} f\left(t-\xi\right)g\left(\xi\right)dt d\xi \\ &= \int_0^\infty g\left(\xi\right)d\xi \int_\xi^\infty e^{-st} f\left(t-\xi\right)dt.\end{aligned}$$

Let $\eta = t - \xi$,

$$\begin{aligned}\int_0^\infty g\left(\xi\right)d\xi \int_\xi^\infty e^{-st} f\left(t-\xi\right)dt &= \int_0^\infty g\left(\xi\right)d\xi \int_0^\infty e^{-s(\eta+\xi)} f\left(\eta\right)d\eta \\ &= \int_0^\infty g\left(\xi\right)e^{-s\xi}d\xi \int_0^\infty f\left(\eta\right)e^{-s\eta}d\eta = \mathcal{L}\left[g(t)\right]\mathcal{L}\left[f(t)\right].\end{aligned}$$

1.4 Laplace Transforms of Derivatives

The Laplace transform of a derivative using equation 1.6 is

$$\mathcal{L}\left[f'\right] = \int_0^\infty e^{-st}\frac{df(t)}{dt}dt = \int_0^\infty e^{-st}d\left(f(t)\right). \tag{1.17}$$

Using integration by parts,

$$\mathcal{L}\left[f'\right] = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)dt = \lim_{t\to\infty} f(t)e^{-st} - \lim_{t\to 0} f(t)e^{-st} + s\mathcal{L}\left[f(t)\right] = -f(0+) + s\mathcal{L}\left[f\right]. \quad (1.18)$$

We have $f(0+)$ instead of $f(0)$, because the limit may not exist at $t = 0$.
Notice that

$$\mathcal{L}\left[f''\right] = \mathcal{L}\left[\left(f'\right)'\right] = -f'(0) + s\mathcal{L}\left[f'\right] = -f'(0) + s\left(-f(0) + s\mathcal{L}\left[f\right]\right) = -f'(0) - sf(0) + s^2\mathcal{L}\left[f\right]. \quad (1.19)$$

By mathematical induction, we can extend this result to obtain the Laplace transform of higher derivatives.

$$\mathcal{L}\left[f^{(n)}\right] = s^n\mathcal{L}\left[f\right] - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0+). \quad (1.20)$$

1.5 Laplace Transforms of Integrals

The Laplace transform of an integral using equation 1.6 is

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)d\tau\right)dt. \quad (1.21)$$

By interchanging orders of integration, we get

$$\begin{aligned}
\int_0^\infty e^{-st}\left(\int_0^t f(\tau)d\tau\right)dt &= \int_0^\infty f(\tau)d\tau \int_\tau^\infty e^{-st}dt && (1.22)\\
&= \frac{1}{s}\int_0^\infty f(\tau)e^{-s\tau}d\tau && (1.23)\\
&= \frac{1}{s}F(s) && (1.24)
\end{aligned}$$

where $Re(s)$ is sufficiently large such that the integrals converge.

1.6 Laplace Transform and Function Behavior

The properties of the initial value, $t = 0+$, and the final-value, $t = \infty$, enables us to find $f(0+)$ and $f(\infty)$ from its Laplace transform.

Theorem 1.4 (Initial-value Theorem)**.**

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \quad (1.25)$$

Proof. We start by taking the limit of equation 1.18

$$\lim_{s\to\infty}\left[sF(s)\right] = \lim_{s\to\infty}\left[f(0+) + \int_0^\infty e^{-st}f'(t)dt\right]. \quad (1.26)$$

When taking limit of the integral as s approaches ∞, $\lim_{s\to\infty} e^{-st}$ is indeterminate for $|t| < \epsilon$ for a small $\epsilon > 0$, so we must integrate on two intervals.

$$\lim_{s\to\infty}\left[\int_0^\infty e^{-st} f'(t)dt\right] = \lim_{s\to\infty}\left\{\lim_{\epsilon\to0+}\left[\int_0^\epsilon e^{-st} f'(t)dt\right] + \lim_{\epsilon\to0+}\left[\int_\epsilon^\infty e^{-st} f'(t)dt\right]\right\} \tag{1.27}$$

Since s is independent of t,

$$\lim_{s\to\infty}\left[\int_0^\infty e^{-st} f'(t)dt\right] = \lim_{s\to\infty}\left\{\lim_{\epsilon\to0+}\left[\int_0^\epsilon f'(t)dt\right] + \int_\epsilon^\infty \lim_{\epsilon\to0+}\left[e^{-st} f'(t)dt\right]\right\} \tag{1.28}$$

$$= \lim_{\epsilon\to0+} f(t)|_0^\epsilon = \lim_{\epsilon\to0+} f(\epsilon) - f(0+). \tag{1.29}$$

Since ϵ is a dummy variable,

$$\lim_{s\to\infty}[sF(s)] = f(0+) + \lim_{t\to0+} f(t) - f(0+) = \lim_{t\to0+} f(t). \tag{1.30}$$

□

Theorem 1.5 (Final-value Theorem)**.**

$$\lim_{t\to\infty} f(t) = \lim_{s\to0} sF(s) \tag{1.31}$$

Proof. We start by taking the limit of equation 1.18,

$$\lim_{s\to0}[sF(s)] = \lim_{s\to0}\left[f(0+) + \int_0^\infty e^{-st} f'(t)dt\right]. \tag{1.32}$$

Since s is independent of t, for the last integral we get

$$\lim_{s\to0}\left[\int_0^\infty e^{-st} f'(t)\right] = \int_0^\infty \lim_{s\to0}\left[e^{-st} f'(t)\right] = \int_0^\infty f'(t)dt.$$

Then,

$$\int_0^\infty f'(t)dt = \lim_{\tau\to\infty}\left[\int_0^\tau f'(t)dt\right] = \lim_{\tau\to\infty}\left[f(t)|_0^\tau\right] = \lim_{\tau\to\infty}\left[f(\tau) - f(0+)\right].$$

Since τ is a dummy variable, we get

$$\lim_{s\to0}[sF(s)] = f(0+) + \lim_{t\to\infty} f(t) - f(0+) = \lim_{t\to\infty} f(t).$$

□

In order for the final-value theorem to hold, all of the poles of $sF(s)$ must have negative real parts to ensure that the function will have terms that decay exponentially. If the poles have positive real parts, the function will have terms that grow exponentially and the function will not be bounded, i.e. $f(\infty)$ does not exist. If the poles are complex conjugates located on the imaginary axis, the function will contain sinusoidal components and $f(\infty)$ is not defined. When the poles are at the origin of the complex plane, the function will contain a constant component, which is the final-value and the steady state of the function.

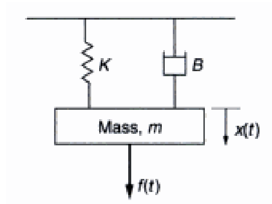


Figure 2: Spring-Mass-Damper System

1.7 Solving Linear Systems

Consider the equation of motion for the spring-mass-damper system,

$$m\ddot{x}(t) + B\dot{x}(t) + Kx(t) = f(t) \tag{1.33}$$

with m, mass, B, damping coefficient, K, spring constant, $x(t)$, displacement, and $f(t)$, external forces. Let $\omega_0 = \sqrt{\frac{K}{m}}$ and $\zeta = \frac{B}{2\sqrt{Km}}$ and make the necessary substitutions in equation 1.33, where we get

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2 x(t) = \frac{f(t)}{m}. \tag{1.34}$$

The substitutions allows us to put the equation in terms of ω_0, natural frequency, and ζ, damping ratio. Take the Laplace transform of equation 1.34,

$$s^2X(s) + 2\zeta\omega_0 sX(s) + \omega_0^2X(s) = \frac{F(s)}{m} + \dot{x}(0) + sx(0) + 2\zeta\omega_0 x(0),$$

where

$$X(s) = \frac{\frac{F(s)}{m} + \dot{x}(0) + sx(0) + 2\zeta\omega_0 x(0)}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

Let $f(t)$ be a step function, $F(s) = \frac{f(0)}{s}$. Using the Final-Value Theorem, we can find the final position of the mass.

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\left(\frac{\frac{f(0)}{ms} + \dot{x}(0) + sx(0) + 2\zeta\omega_0 x(0)}{s^2 + 2\zeta\omega_0 s + \omega_0^2}\right) = \frac{f(0)}{m\omega_0^2}$$

Now, let $x(0) = \dot{x}(0) = 0$, then

$$X(s) = \frac{f(0)}{ms}\left(\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}\right). \tag{1.35}$$

To find the inverse Laplace transform of $X(s)$, to obtain $x(t)$, we begin with partial fraction expansion by finding the roots, λ_1, λ_2, of the characteristic equation, $s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$ where the roots are commonly referred to as the eigenvalues. Notice that we can also write the equation above in terms of the roots,

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = (s - \lambda_1)(s - \lambda_2).$$

Using the quadratic equation,

$$\lambda_{1,2} = \frac{-2\zeta\omega_0 \pm \sqrt{(2\zeta\omega_0)^2 - 4\omega_0^2}}{2} = -\zeta\omega_0 \pm \sqrt{\omega_0^2(\zeta^2 - 1)} = -\omega_0(\zeta \pm \sqrt{\zeta^2 - 1}).$$

We can now write $X(s)$ as

$$X(s) = \frac{f(0)}{m}\left(\frac{A}{s} + \frac{B}{s-\lambda_1} + \frac{C}{s-\lambda_2}\right), \tag{1.36}$$

where A,B, and C can be found using the theorem of residues, and

$$A = \frac{1}{\lambda_1\lambda_2}, B = \frac{1}{\lambda_1(\lambda_1-\lambda_2)}, C = \frac{1}{\lambda_2(\lambda_2-\lambda_1)}.$$

Now we can solve for $x(t)$,

$$x(t) = \frac{f(0)}{m}\left\{A + Be^{\lambda_1 t} + Ce^{\lambda_2 t}\right\} = \frac{f(0)}{m}\left\{\frac{1}{\lambda_1\lambda_2} + \frac{e^{\lambda_1 t}}{\lambda_1(\lambda_1-\lambda_2)} + \frac{e^{\lambda_2 t}}{\lambda_2(\lambda_2-\lambda_1)}\right\}.$$

2 Control Systems Theory

There are two types of Control Theory: classical and modern. The Classical Control theory involves studying single-input single-output, linear, time-invariant systems analyzed in the frequency-doman analysis utilizing the Laplace Transform. The Modern Control theory involves studying multiple-input multiple-output, non-linear, time-varying systems in the time domain through use of the state-space equations. A time-varying system has one or more parameters that may vary as a function of time.

Continuous time-invariant

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \tag{2.1}$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) \tag{2.2}$$

Continuous time-variant

$$\dot{x}(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t) \tag{2.3}$$

$$y(t) = \mathbf{C}(t)x(t) + \mathbf{D}(t)u(t) \tag{2.4}$$

Control systems are categorized into two basic types: open-loop control systems and closed-loop control systems. An **open-loop control system** is a system in which the output has no effect on the input signal, whereas, in a **closed-loop control system** the output has an effect on the input quantity in order to maintain the desired output system by controlling the states or outputs of the system. In essence, the closed-loop control system implements a a feedback loop that returns the output back to the controller so that the system can self-correct in an effort to maintain the desired output, such as the cruise control on an automobile or a thermostat in a house. Since the open-loop system does not utilize a feedback system, the system cannot correct any system errors or compensate for disturbances in the system. An open-loop system is mostly used for systems where the relationship between the input and resultant state are modeled by a mathematical formula such as the spring-mass-damper system.

An important aspect of a classical control system is its transfer function, which fully describes the system. The transfer function, $H(s)$, is the ratio of the output and input of the system and is represented in the frequency-domain as a Laplace transform,

$$H(s) = \frac{Y(s)}{U(s)} \tag{2.5}$$

where $Y(s)$ and $U(s)$, is the output and the input, both expressed in the frequency-domain. The inverse Laplace Transform of the transfer function, $h(t)$, is the **impulse response**, which represents the response of the system if the input $u(t)$ is equal to the $dirac - \delta(t)$ function. The transfer function of the continuous system

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t)$$

can also be expressed as

$$H(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}. \tag{2.6}$$

With the transfer function, we can easily find the output of the system since $U(s)H(s) = Y(s)$.

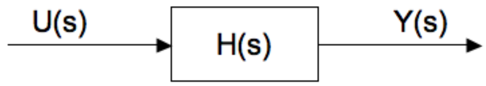


Figure 3: Simple Open-loop Control System

Consider the spring-mass-damper equation,

$$s^2X(s) + 2\zeta\omega_0 sX(s) + \omega_0^2 X(s) = \frac{F(s)}{m}$$

where $x(0) = \dot{x}(0) = 0$. Then, the transfer function is

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{m(s^2 + 2\zeta\omega_0 s + \omega_0^2)}. \tag{2.7}$$

The closed-loop control system is more complex than an open-loop control system due to the feedback element, **K**(s), where the system is represented, in the frequency domain, as in figure 4.

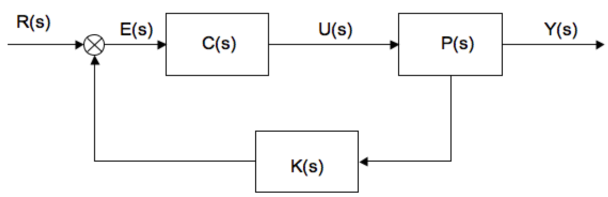


Figure 4: Simple Closed-loop Control System

1. **R**(s) - Desired Output
2. **C**(s) - Controller
3. **U**(s) - System input, **U**(s) = **C**(s)**E**(s)
4. **P**(s) - Plant (System)
5. **K**(s) - Feedback (Sensor)
6. **Y**(s) - Measured Output
7. **E**(s) - Error or actuating signal, **E**(s) = **R**(s) - **K**(s)**Y**(s)

Since,

$$Y(s) = \frac{P(s)C(s)}{1 + K(s)P(s)C(s)} R(s) = H(s)R(s) \tag{2.8}$$

where **H**(s) is the **closed-loop transfer function**, we can simplify the system to figure 5.

Note that **G**(s) = **P**(s)**C**(s) is the **forward, open-loop, transfer function**, where the system can also be simplified to figure 6.

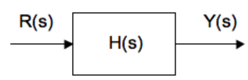


Figure 5: Reduced Feedback System

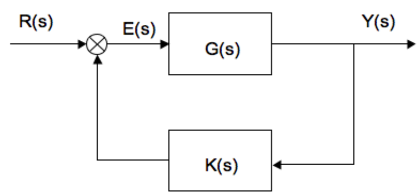


Figure 6: Simplified Feedback System

2.1 System Properties

The primary goal of many, if not all, control systems is to improve and/or optimize the performance of a dynamical system. There are three main properties of a system: stability, controllability, and observability.

2.1.1 Controllability

Controllability is an indicator for if it is possible to alter states from an initial value to a final value within a finite time interval, in essence, the ability to control the states of the system.

Definition 2.1. *A system is* ***state controllable*** *if, by applying a proper input, $u(t)$, the state can be changed from any given state in a finite amount of time.*

Theorem 2.2. *A continuous time-invariant state-space model is controllable if and only if $rank(\mathcal{C}) = n$, or full rank, where n is the order of the system and the controllability matrix $\mathcal{C}$ is defined as*

$$\mathcal{C} = \begin{bmatrix} b & Ab & . & . & . & A^{n-1}b \end{bmatrix}. \tag{2.9}$$

Using the spring-mass-damper system, we can start by putting the state-space equations and system output equations in matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K}{m} & \frac{-B}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t) \tag{2.10}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.11}$$

and

$$\mathcal{C} = \left[\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \begin{bmatrix} \frac{1}{m} \\ \frac{-B}{m^2} \end{bmatrix} \begin{bmatrix} \frac{-B}{m^2} \\ \frac{B^2-Km}{m^3} \end{bmatrix} \right] = \begin{bmatrix} 0 & \frac{1}{m} & \frac{-B}{m^2} \\ \frac{1}{m} & \frac{-B}{m^2} & \frac{B^2-Km}{m^3} \end{bmatrix}, \tag{2.12}$$

where $rank(\mathcal{C}) = 2$. Since the system is of second-order, the spring-mass-damper system is controllable.

2.1.2 Observability

Observability is an indicator of the ability to alter the state of the system from an initial state to a final state over a finite time interval through use of an appropriate control input, so it is the system's ability to supply the information necessary to estimate all the states of the system.

Definition 2.3. *A system is* ***observable*** *if its state at some time* t_0 *can be determined from the values of the system's output over a finite time interval* $[t_0, t_f]$.

Theorem 2.4. *A continuous time-invariant state-space model is observable if and only if rank(*$\mathcal{O}$*)* $= n$*, where* n *is the order of the system and the observability matrix* $\mathcal{O}$ *is defined as*

$$\mathcal{O} = \begin{bmatrix} c \\ cA \\ . \\ . \\ . \\ cA^{n-1} \end{bmatrix} \tag{2.13}$$

Intuitively, we know that the spring-mass-damper system is observable, so we will verify using the observability matrix. Using equations 2.10 and 2.11, we get that

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ \frac{-K}{m} & \frac{-B}{m} \\ \frac{BK}{m^2} & \frac{B^2}{m^2} - \frac{K}{m} \end{bmatrix}, \tag{2.14}$$

where $rank(\mathcal{O}) = 2$, so the spring-mass-damper system is observable, as expected.

2.1.3 Stability

Stability is the most important and necessary property of a dynamic system primarily due to safety issues within a system, such as the control system that keeps a missile from hitting arbitrary targets instead of the intended target. In order for the system to be stable, the eigenvalues of the state matrix must have negative real parts, otherwise, the system will be sinusoidal, saturated, or become unstable. The stability of a time-invariant state-space model can also be determined by studying the transfer function, $H(s)$, of the system, where we can use the poles of the function to determine the existence of stability and/or the type of stability. We can express the transfer function as

$$H(s) = \frac{(s - z_0)..(s - z_i)..(s - z_m)}{(s - p_0)...(s - p_i)..(s - p_n)}, \tag{2.15}$$

where z_i and p_i are the zeros and poles of the function, respectively, which leads us to the stability criteria:

The spring-mass-damper system is stable since it is physically unable to produce an output whose amplitude increases as $t \to \infty$. Without the damper, $B = 0$, if we simply push the mass once, $f(t) = \delta(t)$, the mass will oscillate forever with constant amplitude and frequency represented by a sinusoidal solution. However, if we add the damper, the mass will oscillate at first, but the amplitude will decrease until the

Table 1: Stability Criteria

Position of Pole(s)	Form of Response	Behaviour
Negative Real	Ae^{-bt}	Damped exponential
Complex Conjugates with negative real parts	$Ae^{-bt}\sin(ct+\phi)$	Exponential damped sinusoid
Poles located at the origin	A	Constant
Complex conjugates located on the imaginary axis	$A\sin(ct+\phi)$	Constant sinusoid
Positive real	Ae^{bt}	Increasing exponential (unstable)
Complex conjugates with positive real parts	$Ae^{bt}\sin(ct+\phi)$	Exponentially increasing sinusoid (unstable)

mass comes to rest. Either situation can be indicated by the location of the poles in the frequency domain. The spring-mass system equation is

$$\ddot{x}(t)+\omega_0^2 x(t)=\frac{f(t)}{m} \tag{2.16}$$

with the transfer function

$$s^2X(s)+\omega_0^2X(s)=\frac{F(s)}{m}\Rightarrow H(s)=\frac{X(s)}{F(s)}=\frac{1}{m(s^2+\omega_0^2)}. \tag{2.17}$$

By finding the roots of the characteristic equation, $s^2+\omega_0^2=0$, we find that poles are located at $p_{1,2}=\pm i\omega_0$, which indicates, as expected, that the solution will oscillate. Adding back the damper,

$$H(s)=\frac{X(s)}{F(s)}=\frac{1}{m(s^2+2\zeta\omega_0 s+\omega_0^2)} \tag{2.18}$$

and

$$\lambda_{1,2}=p_{1,2}=-\omega_0(\zeta\pm\sqrt{\zeta^2-1}) \tag{2.19}$$

The poles are either negative, $Re(\lambda)<0$, indicating the system converges, or equal to 0, where the system will be marginally stable so in order for the system to be stable, without oscillations, ζ and ω_0 must be strictly positive, which is understandable, since physically, the spring-mass-damper system cannot have negative friction or negative damping coefficient.

2.2 Control System Design

2.2.1 Proportional-Integral-Derivative (PID)

The **PID controller** is the most commonly used feedback control design, where

$$u(t)=K_Pe(t)+K_I\int e(t)dt+K_D\frac{d}{dt}e(t). \tag{2.20}$$

By adjusting the tuning parameters, K_P, K_I, and K_D, sometimes iteratively, we can obtain the "correct" system dynamics. $K_Pe(t)$ is the proportional term where K_P is the proportional gain. The proportional term produces a large change in the output for a given error disturbance sometimes resulting in instability if the gain is too high, on the other hand, a small gain can reduce the sensitivity of the controller. $K_I \int_0^t e(\tau)d\tau$ is the integral term where K_I is the integral gain. The integral term causes the system to stabilize to the desired output and minimizes the steady-state error produced by the proportional term, but it can also cause the system to overshoot the desired value. $K_D \frac{d}{dt} e(t)$ is the derivative term where K_D is the derivative gain. The derivative term reduces the magnitude of the overshoot from the integral term and improves stability, but it is highly sensitive to noise in the error term causing instability for a large derivative gain and noise. The Laplace transform of the PID Controller is

$$U(s) = K_P E(s) + K_I \frac{E(s)}{s} + sK_D E(s) = \left(K_P + \frac{K_I}{s} + sK_D\right) E(s) \tag{2.21}$$

where the controller transfer function is

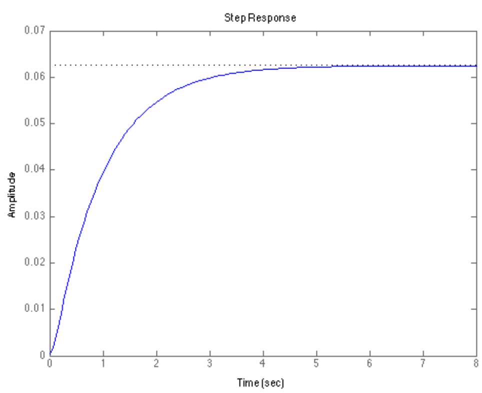
$$C(s) = \frac{U(s)}{E(s)} = \left(K_P + \frac{K_I}{s} + sK_D\right) \tag{2.22}$$

Table 2: Effects of Inscreasing Parameters

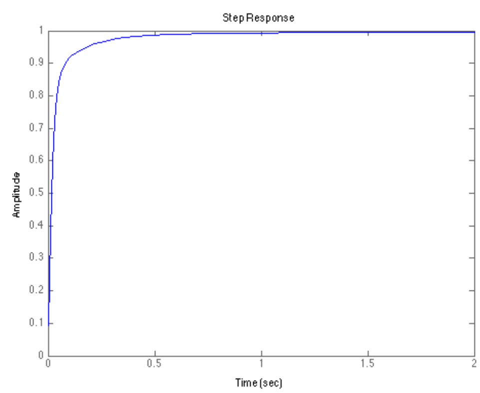
Parameter	Rise Time	Overshoot	Settling time	Error at Equilibrium
K_P	Decrease	Increase	Small change	Decrease
K_I	Decrease	Increase	Increase	Eliminate
K_D	Indefinite	Decrease	Decrease	None

Example 2.5. *For this example, we will control the spring-mass-damper system, equation 1.34, using the PID controller. Let $\zeta = 2$ and $\omega_0 = 4$, then the step-response of the system can be seen in figure 8(a). Let $K_P = 450$, $K_I = 300$, and $K_D = 50$.*

Notice in figure 7(b) that the system still stabilizes, but does so quicker than the system without the controller and stabilizes to a different value. However, it takes tuning to find the right combination of K_I, K_P, and K_D to get the desired results, because sometimes you may only need one or two of the three, such as in figure 8(b), figure 8(c), and figure 8(d). In figure 8(b), we set $K_D = 0$ causing the system to overshoot the equilibrium, which is different from the original equilibrium, before settling. In figure 8(c), we set $K_P = 0$ increasing the settling time and causing the system to overshoot. In figure 8(d), we set $K_I = 0$, which has similar behaviour to the original PID controller except it settles to a slight lower value.

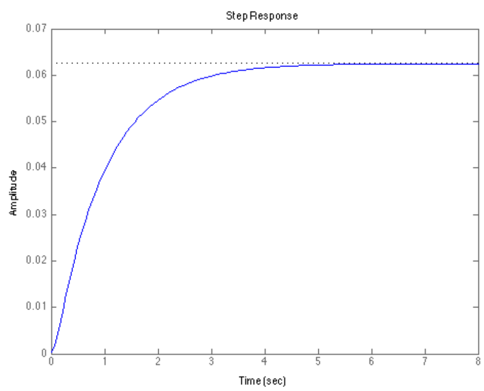


(a) No PID

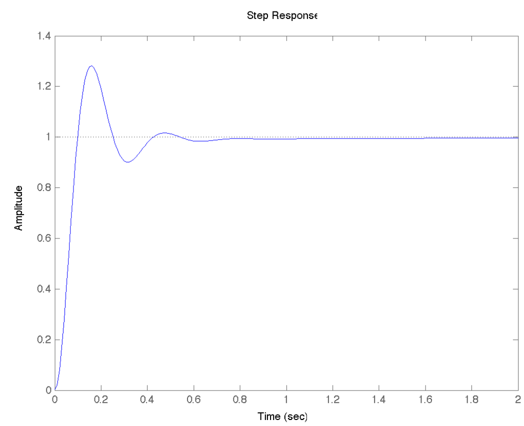


(b) With PID

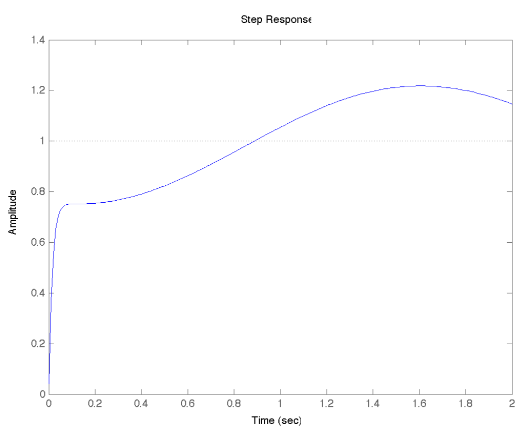
Figure 7: Spring-Mass-Damper System



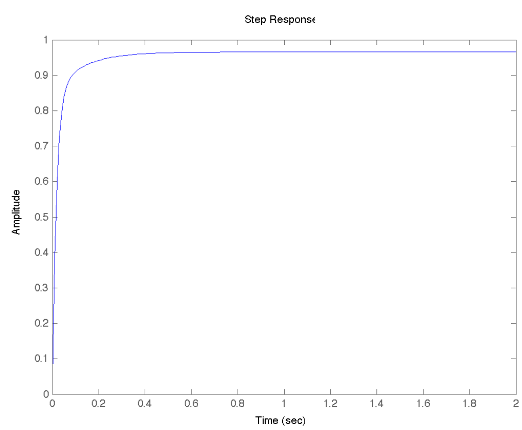
(a) No PID



(b) Proportional-Integral



(c) Integral-Derivative



(d) Proportional-Derivative

Figure 8: Spring-Mass-Damper System with PID Controller

2.2.2 Pole Placement

If a closed system is completely controllable, its poles can be arbitrarily relocated using the state-variable feedback where $u(t) = r(t) + \mathbf{K}x(t)$ to improve the stability of the system, where we get the following system:

$$\dot{x}(t) = (\mathbf{A} + \mathbf{BK})x(t) + \mathbf{B}r(t) \tag{2.23}$$
$$y(t) = \mathbf{C}x(t) \tag{2.24}$$

where $r(t)$ is the input vector, $\mathbf{K}$ is the state-feedback, or gain, matrix, and $\mathbf{A} + \mathbf{BK}$ is the system matrix. Taking the Laplace transform,

$$sX(s) = (\mathbf{A} + \mathbf{BK})X(s) + \mathbf{B}R(s) \Rightarrow X(s) = \frac{\mathbf{B}R(s)}{s\mathbf{I} - (\mathbf{A} + \mathbf{BK})} \tag{2.25}$$
$$Y(s) = \mathbf{C}X(s) \tag{2.26}$$

and

$$Y(s) = \mathbf{C}\left(\frac{\mathbf{B}}{s\mathbf{I} - (\mathbf{A} + \mathbf{BK})}\right)R(s) \tag{2.27}$$
$$\tag{2.28}$$

where the transfer function is

$$\frac{Y(s)}{R(s)} = \frac{\mathbf{CB}}{s\mathbf{I} - (\mathbf{A} + \mathbf{BK})}$$

So, the eigenvalues of the characteristic equation, $|s\mathbf{I} - (\mathbf{A} + \mathbf{BK})| = 0$, can be changed by setting the values of $\mathbf{K}$.

Example 2.6. *Consider the second-order system*

$$\dot{x}(t) = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{2.29}$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \tag{2.30}$$

Let $u(t) = r(t) + \boldsymbol{K}x(t)$, then

$$\dot{x}(t) = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(r(t) + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} x(t)\right) \tag{2.31}$$
$$= \begin{bmatrix} -1 + k_1 & 3 + k_2 \\ k_1 & -2 + k_2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r(t) \tag{2.32}$$

So our feedback system is,

$$\dot{x}(t) = \begin{bmatrix} -1 + k_1 & 3 + k_2 \\ k_1 & -2 + k_2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r(t) \tag{2.33}$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \tag{2.34}$$

Taking the Laplace Transform,

$$sX(s) = \begin{bmatrix} -1+k_1 & 3+k_2 \\ k_1 & -2+k_2 \end{bmatrix} X(s) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} R(s) \tag{2.35}$$

$$Y(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} X(s) \tag{2.36}$$

and finding the overall transfer function,

$$\frac{Y(s)}{R(s)} = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix}}{\begin{bmatrix} s+1-k_1 & -3-k_2 \\ -k_1 & s+2-k_2 \end{bmatrix}} \tag{2.37}$$

we find that the characteristic equation is:

$$|s\boldsymbol{I} - (\boldsymbol{A} + \boldsymbol{BK})| = \begin{vmatrix} s+1-k_1 & -3-k_2 \\ -k_1 & s+2-k_2 \end{vmatrix} = s^2 + (3-k_1-k_2)s + (2-5k_1-k_2) \tag{2.38}$$

Without the feedback, $k_1 = k_2 = 0$, the eigenvalues are $\lambda_{1,2} = -1, -2$. If we want to move the eigenvalues to $\lambda_{1,2} = -5, -6$, the characteristic equation would need to be $s^2 + 11s + 30$, so equating the coefficients

$$3 - k_1 - k_2 = 11 \tag{2.39}$$

$$2 - 5k_1 - k_2 = 30 \tag{2.40}$$

we find that $\boldsymbol{K} = \begin{bmatrix} -5 & -3 \end{bmatrix}$.

3 The Gyro-pendulum

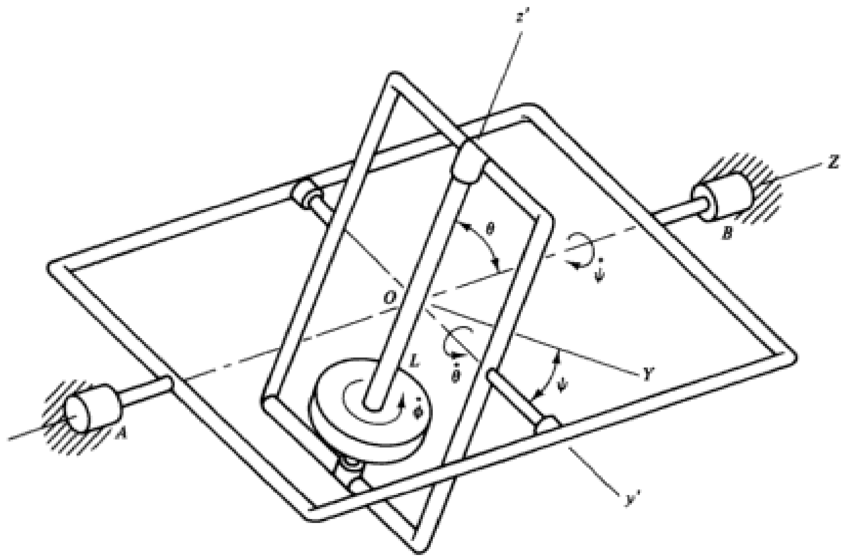


Figure 9: Gyro-pendulum [5]

The gyro-pendulum modeled in figure 9 is often used in intertial guidance applications to locate the vertical direction for a vehicle moving around the earth.
We will begin with the equations of motion for the gyro-pendulum and find the solutions for the linear control system using Laplace Transform. We can find the equations of motion through use of the Lagrange equations:

$$\mathcal{Q}_j = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\mathcal{L}}{\partial q_j} \tag{3.1}$$

where $\mathcal{L} = \mathcal{T} - \mathcal{V}$, $\mathcal{V}$ is the potential energy, $\mathcal{T}$ is the kinetic energy, and q_j are the generalized coordinates, which are ψ, ϕ, and θ, in this case. We have the following definitions: $\dot{\phi}$ is spin rate, m is mass, g is gravity, θ(t) is nutation response, ψ(t) is precessional response, ω is angular speed, $I_{x'x'}$,$I_{y'y'}$, $I_{z'z'}$ are centroidal principal moments of inertia of the flywheel parallel and transverse to the spin axis, R is radius of the servometer, and L is distance from the origin to the center of mass of the servometer. Assume that the spin rate is constant, friction is negligible, and we will ignore the inertia of the gimbals. In order to find the Lagrange equations,

$$\frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right\} - \frac{\partial \mathcal{L}}{\partial \theta} = \mathcal{Q}_\theta$$

$$\frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right\} - \frac{\partial \mathcal{L}}{\partial \psi} = \mathcal{Q}_\psi$$

we must find the equations for kinetic and potential energy. From the figure, we gather that

$$\overline{w} = \dot{\psi} Z + \dot{\theta} y' + \dot{\phi} z' = \dot{\psi}\left(-\sin\theta x' + \cos\theta z'\right) + \dot{\theta} y' + \dot{\phi} z' \tag{3.2}$$
$$\tag{3.3}$$

and

$$I_{z'z'} = \frac{1}{2} mR^2 = \hat{I} \tag{3.4}$$
$$I_{x'x'} = I_{y'y'} = \frac{1}{4} mR^2 + mL^2 = I, \tag{3.5}$$

due to axisymmetry. Kinetic energy is defined as:

$$\mathcal{T} = \frac{1}{2} I_{x'x'} \omega_{x'}^2 + \frac{1}{2} I_{y'y'} \omega_{y'}^2 + \frac{1}{2} I_{z'z'} \omega_{z'}^2,$$

and plugging in the angular speed and moments of intertia, we get

$$\mathcal{T} = \frac{1}{2} I \left(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta\right) + \frac{1}{2} \hat{I} \left(\dot{\psi}\cos\theta + \dot{\phi}\right)^2.$$

Potential energy is defined as:

$$\mathcal{V} = mgh,$$

where h is the height, thus, the potential energy for the gyropendulum is

$$\mathcal{V} = mg\left(-Lz' \cdot X\right) = -mgL\sin\theta\cos\psi.$$

We will assume that the only external forces are the initial pushes on the ψ-angle and θ-angle, thus $Q_\psi = Q_\theta = \delta(t)$. Applying the Lagrange equations,

$$\begin{aligned}
Q_\psi &= \frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right\} - \frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt}\left\{\frac{\partial\left(\mathcal{T}-\mathcal{V}\right)}{\partial \dot{\psi}}\right\} - \frac{\partial\left(\mathcal{T}-\mathcal{V}\right)}{\partial \psi} = \frac{d}{dt}\left\{\frac{\partial \mathcal{T}}{\partial \dot{\psi}}\right\} - \frac{d}{dt}\left\{\frac{\partial \mathcal{V}}{\partial \dot{\psi}}\right\} - \frac{\partial \mathcal{T}}{\partial \psi} + \frac{\partial \mathcal{V}}{\partial \psi} \\
&= \ddot{\psi}\left(\hat{I}\cos^2\theta + I\sin^2\theta\right) + 2\left(I - \hat{I}\right)\dot{\psi}\dot{\theta}\sin\theta\cos\theta - \hat{I}\dot{\theta}\dot{\phi}\sin\theta + mgL\sin\theta\sin\psi
\end{aligned}$$

$$\begin{aligned}
Q_\theta &= \frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right\} - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left\{\frac{\partial\left(\mathcal{T}-\mathcal{V}\right)}{\partial \dot{\theta}}\right\} - \frac{\partial\left(\mathcal{T}-\mathcal{V}\right)}{\partial \theta} = \frac{d}{dt}\left\{\frac{\partial \mathcal{T}}{\partial \dot{\theta}}\right\} - \frac{d}{dt}\left\{\frac{\partial \mathcal{V}}{\partial \dot{\theta}}\right\} - \frac{\partial \mathcal{T}}{\partial \theta} + \frac{\partial \mathcal{V}}{\partial \theta} \\
&= I\ddot{\theta} + \hat{I}\left(\dot{\psi}\cos\theta + \dot{\phi}\right)\left(-\dot{\psi}\sin\theta\right) + I\left(\dot{\psi}^2\sin\theta\cos\theta\right) - mgL\cos\theta\cos\psi \\
&= I\ddot{\theta} + \left(\hat{I} - I\right)\dot{\psi}^2\sin\theta\cos\theta + \hat{I}\dot{\psi}\dot{\phi}\sin\theta - mgL\cos\theta\cos\psi
\end{aligned}$$

We find that the equations of motions are:

$$\ddot{\psi}\left(\hat{I}\cos^2\theta + I\sin^2\theta\right) + 2\left(I - \hat{I}\right)\dot{\psi}\dot{\theta}\sin\theta\cos\theta - \hat{I}\dot{\phi}\dot{\theta}\sin\theta + mgL\sin\theta\sin\psi = \delta(t) \tag{3.6}$$

$$I\ddot{\theta} + \left(\hat{I} - I\right)\dot{\psi}^2 \sin\theta\cos\theta + \hat{I}\dot{\psi}\dot{\phi}\sin\theta - mgL\cos\theta\cos\psi = \delta(t). \tag{3.7}$$

where $\hat{I} = I_{z'z'}$, $I = I_{x'x'}, I_{y'y'}$ and $\delta(t)$ indicates that the ψ-angle and θ-angle are given an initial push. We can linearize the problem by assuming that $\psi << 1$ and $\theta = \frac{\pi}{2} - \tau$ such that $\tau << 1 \Rightarrow \dot{\theta} = -\dot{\tau}$, which leads to: $\cos\psi \approx 1$, $\cos\theta \approx \tau$, $\sin\psi \approx \psi$, and $\sin\theta \approx 1$. Then equations 3.6 and 3.7 can be reduced to

$$I\ddot{\psi} + \hat{I}\dot{\phi}\dot{\tau} + mgL\psi = \delta(t) \tag{3.8}$$

$$-I\ddot{\tau} + \hat{I}\dot{\phi}\dot{\psi} - mgL\tau = \delta(t) \tag{3.9}$$

Let $\sigma = \frac{\hat{I}}{I}\dot{\phi}$ and $\omega^2 = \frac{mgL}{I}$, then

$$\ddot{\psi} + \sigma\dot{\tau} + \omega^2\psi = u(t) \tag{3.10}$$

and

$$-\ddot{\tau} + \sigma\dot{\psi} - \omega^2\tau = u(t) \tag{3.11}$$

where $u(t) = \frac{\delta(t)}{I} \approx \delta(t)$.

3.1 Controllability and Observability

We can convert equations 3.10 and 3.11 to a system of first-order differential equations:

$$\dot{x}_1 = x_3 \tag{3.12}$$

$$\dot{x}_2 = x_4 \tag{3.13}$$

$$\dot{x}_3 = -\sigma x_4 - \omega^2 x_1 + u(t) \tag{3.14}$$

$$\dot{x}_4 = \sigma x_3 - \omega^2 x_2 + u(t) \tag{3.15}$$

so we are able to show the system in matrix form:

$$\dot{\vec{x}}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega^2 & 0 & 0 & -\sigma \\ 0 & -\omega^2 & \sigma & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u(t) \tag{3.16}$$

$$\vec{y}(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \vec{x}(t) \tag{3.17}$$

The controllability and observability matrix are

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & -\sigma & -\omega^2 - \sigma^2 \\ 0 & 1 & \sigma & -\omega^2 - \sigma^2 \\ 1 & -\sigma & -\omega^2 - \sigma^2 & \omega^2\sigma + \sigma(\omega^2 + \sigma^2) \\ 1 & \sigma & -\omega^2 - \sigma^2 & -\omega^2\sigma - \sigma(\omega^2 + \sigma^2) \end{bmatrix} \tag{3.18}$$

$$\mathcal{O} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -\omega^2 & -\omega^2 & \sigma & -\sigma \\ -\omega^2\sigma & \omega^2\sigma & -\omega^2 - \sigma^2 & -\omega^2 - \sigma^2 \end{bmatrix} \tag{3.19}$$

where rank($\mathcal{C}$) = 4 and rank($\mathcal{O}$) = 4, so the system is observable and controllable. To find stability, we take the Laplace Transform of equations 3.16 and 3.17 and find the transfer function,

$$\frac{Y(s)}{U(s)} = \frac{1}{\begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ \omega^2 & 0 & s & \sigma \\ 0 & \omega^2 & -\sigma & s \end{bmatrix}} \tag{3.20}$$

Then the characteristic equation is $s^4 + (2\omega^2 + \sigma^2)s^2 + \omega^4 = 0$, where the eigenvalues are purely imaginary, $Re(\lambda_i) = 0$.

$$\lambda_1 = -\left(\frac{-\sigma}{2}\sqrt{4\omega^2+\sigma^2} - \frac{2\omega^2+\sigma^2}{2}\right)^{\frac{1}{2}} \tag{3.21}$$

$$\lambda_2 = \left(\frac{-\sigma}{2}\sqrt{4\omega^2+\sigma^2} - \frac{2\omega^2+\sigma^2}{2}\right)^{\frac{1}{2}} \tag{3.22}$$

$$\lambda_3 = -\left(\frac{\sigma}{2}\sqrt{4\omega^2+\sigma^2} - \frac{2\omega^2+\sigma^2}{2}\right)^{\frac{1}{2}} \tag{3.23}$$

$$\lambda_4 = \left(\frac{\sigma}{2}\sqrt{4\omega^2+\sigma^2} - \frac{2\omega^2+\sigma^2}{2}\right)^{\frac{1}{2}} \tag{3.24}$$

For all $\omega > 0$ and $\sigma > 0$, the eigenvalues are complex pairs located on the imaginary axis, so the system is marginally stable indicating the output will oscillate indefinitely.

3.2 Stability and Solution Using Laplace Transform

To find the solution using Laplace Transform, we will begin with

$$(s^2 + \omega^2)\Psi(s) + s\sigma T(s) = 1 \tag{3.25}$$

$$-(s^2 + \omega^2)T(s) + s\sigma\Psi(s) = 1 \tag{3.26}$$

which are the Laplace Transform of equations 3.10 and 3.11, and we will solve for $\Psi(s)$ and $T(s)$.

$$\begin{bmatrix} s^2+\omega^2 & s\sigma \\ s\sigma & -s^2-\omega^2 \end{bmatrix}\begin{bmatrix} \Psi \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.27}$$

$$\begin{bmatrix} \Psi \\ T \end{bmatrix} = \begin{bmatrix} s^2+\omega^2 & s\sigma \\ s\sigma & -s^2-\omega^2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{s\sigma+(\omega^2+s^2)}{(s^2+\omega^2)^2+(s\sigma)^2} \\ \frac{s\sigma-(\omega^2+s^2)}{(s^2+\omega^2)^2+(s\sigma)^2} \end{bmatrix} \tag{3.28}$$

Using Partial Fraction expansion and Theorem of Residues, we can find

$$T(s) = \frac{r_1^{(1)}}{s-\lambda_1} + \frac{r_2^{(1)}}{s-\lambda_2} + \frac{r_3^{(1)}}{s-\lambda_3} + \frac{r_4^{(1)}}{s-\lambda_4} \tag{3.29}$$

$$\Psi(s) = \frac{r_1^{(2)}}{s-\lambda_1} + \frac{r_2^{(2)}}{s-\lambda_2} + \frac{r_3^{(2)}}{s-\lambda_3} + \frac{r_4^{(2)}}{s-\lambda_4} \tag{3.30}$$

$$\tag{3.31}$$

where λ_n and r_n, are the eigenvalues and residues, respectively. Let $(s^2+\omega^2)^2+(s\sigma)^2=0$ and notice that this is the same characteristic equation from the transfer function, so the eigenvalues are also the same, but for simplicity, we will still refer to them as $\lambda's$. The residues will be of the form

$$r_a^{(1)} = \frac{\lambda_a\sigma - (\omega^2+\lambda_a^2)}{(\lambda_a-\lambda_b)(\lambda_a-\lambda_c)(\lambda_a-\lambda_d)} \tag{3.32}$$

$$r_a^{(2)} = \frac{\lambda_a\sigma + (\omega^2+\lambda_a^2)}{(\lambda_a-\lambda_b)(\lambda_a-\lambda_c)(\lambda_a-\lambda_d)} \tag{3.33}$$

Then,

$$\mathcal{L}^{-1}\left\{\frac{r_a}{s-\lambda_a}\right\} = r_a\mathcal{L}^{-1}\left\{\frac{1}{s-\lambda_a}\right\} = r_a e^{\lambda_a t}, \tag{3.34}$$

which leads to the solutions

$$\tau(t) = \sum_{i=1}^{4} r_i^{(1)} e^{\lambda_i t} \tag{3.35}$$

$$\psi(t) = \sum_{i=1}^{4} r_i^{(2)} e^{\lambda_i t} \tag{3.36}$$

where

$$\theta(t) = \frac{\pi}{2} - \sum_{i=1}^{4} r_i^{(1)} e^{\lambda_i t} \tag{3.37}$$

By plotting the solutions,

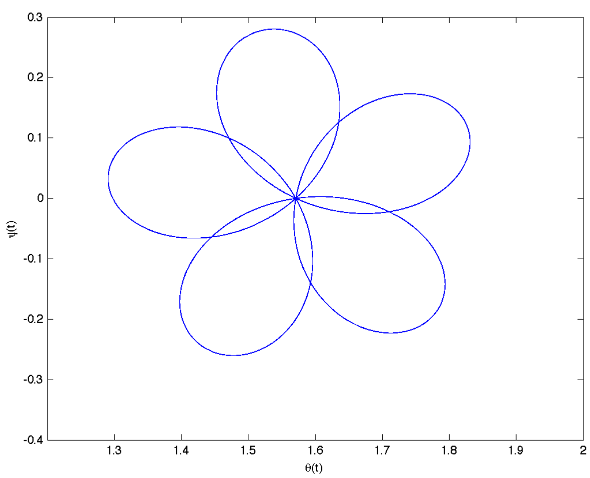


Figure 10: ψ(t) vs. θ(t)

we can see that the solutions, in fact, oscillate and pass through an equilibrium point at $(\frac{\pi}{2}, 0)$.

3.3 Pole Placement

Since the system is controllable, we can move the poles of the transfer function using pole placement. Let $\vec{u}(t) = \mathbf{K}\vec{x}(t) + \vec{r}(t)$, then our system is

$$\dot{\vec{x}}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_1 - \omega^2 & k_2 & k_3 & k_4 - \sigma \\ k_1 & k_2 - \omega^2 & k_3 + \sigma & k_4 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} r(t) \tag{3.38}$$

$$\vec{y}(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \vec{x}(t) \tag{3.39}$$

and the transfer function is

$$\frac{Y(s)}{U(s)} = \frac{1}{\begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ -k_1 + \omega^2 & -k_2 & s - k_3 & -k_4 + \sigma \\ -k_1 & -k_2 + \omega^2 & -k_3 - \sigma & -k_4 + s \end{bmatrix}} \tag{3.40}$$

The characteristic equation is

$$s^4 - (k_3 + k_4)s^3 + (2\omega^2 + \sigma^2 - k_1 - k_2 + k_3\sigma - k_4\sigma)s^2 + (k_1\sigma - k_2\sigma - k_3\omega^2 - k_4\omega^2)s - k_1\omega^2 - k_2\omega^2 + \omega^4 = 0. \tag{3.41}$$

Assume we want to place the poles at s = $-1 \pm 4i, -2, -1$ then our characteristic equation should be

$$s^4 + 5s^3 + 25s^2 + 55s + 34 = 0. \tag{3.42}$$

Equating the coefficients, we get

$$k_1 = \frac{\sigma\omega^4 - 34\sigma - 5\omega^4 + 55\omega^2}{2\omega^2\sigma} \tag{3.43}$$

$$k_2 = \frac{\sigma\omega^4 - 34\sigma + 5\omega^4 - 55\omega^2}{2\omega^2\sigma} \tag{3.44}$$

$$k_3 = \frac{-\sigma^2\omega^2 - 5\sigma\omega^2 - \omega^4 + 25\omega^2 - 34}{2\omega^2\sigma} \tag{3.45}$$

$$k_4 = \frac{\sigma^2\omega^2 - 5\sigma\omega^2 + \omega^4 - 25\omega^2 + 34}{2\omega^2\sigma} \tag{3.46}$$

Assume $\sigma = 3$ and $\omega = 2$, then $\mathbf{K} = \begin{bmatrix} \frac{43}{12} & \frac{-97}{12} & \frac{-23}{12} & \frac{-37}{12} \end{bmatrix}$ and our characteristic equation becomes $s^4 + 5s^3 + 25s^2 + 55^2 + 34 = 0$.

3.4 Solution with Pole Placement

Continuing from the previous section, we will find the solution using our new eigenvalues with the σ and ω given, by converting our system equations back to the equations of motion.

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_1 - \omega^2 & k_2 & k_3 & k_4 - \sigma \\ k_1 & k_2 - \omega^2 & k_3 + \sigma & k_4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u(t) \tag{3.47}$$

$$\ddot{\psi}(t) = \left(k_1 - \omega^2\right)\psi(t) + k_2\tau(t) + k_3\dot{\psi}(t) + (k_4 - \sigma)\,\dot{\tau}(t) + \delta(t) \tag{3.48}$$

$$\ddot{\tau}(t) = k_1\psi(t) + \left(k_2 - \omega^2\right)\tau(t) + (k_3 + \sigma)\,\dot{\psi}(t) + k_4\dot{\tau}(t) + \delta(t) \tag{3.49}$$

Taking the Laplace transform,

$$s^2\Psi(s) = \left(k_1 - \omega^2\right)\Psi(s) + k_2T(s) + k_3s\Psi(s) + (k_4 - \sigma)\,sT(s) + 1 \tag{3.50}$$

$$s^2T(s) = k_1\Psi(s) + \left(k_2 - \omega^2\right)T(s) + (k_3 + \sigma)\,s\Psi(s) + k_4sT(s) + 1 \tag{3.51}$$

Substituting the values and solving the system,

$$\left(12s^2 + 23s + 5\right)\Psi(s) + (73s + 97)\,T(s) = 12 \tag{3.52}$$

$$-\left(13s + 43\right)\Psi(s) + \left(12s^2 + 37s + 145\right)T(s) = 12 \tag{3.53}$$

$$\begin{bmatrix} \left(12s^2 + 23s + 5\right) & (73s + 97) \\ -\left(13s + 43\right) & \left(12s^2 + 37s + 145\right) \end{bmatrix} \begin{bmatrix} \Psi \\ T \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \end{bmatrix} \tag{3.54}$$

$$\begin{bmatrix} \Psi \\ T \end{bmatrix} = \begin{bmatrix} \left(12s^2 + 23s + 5\right) & (73s + 97) \\ -\left(13s + 43\right) & \left(12s^2 + 37s + 145\right) \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 12 \end{bmatrix} \tag{3.55}$$

The solution in the frequency domain is

$$\Psi(s) = \frac{s^2 - 3s + 4}{s^4 + 5s^3 + 25s^2 + 55^2 + 34} \tag{3.56}$$

$$T(s) = \frac{s^2 + 3s + 4}{s^4 + 5s^3 + 25s^2 + 55^2 + 34} \tag{3.57}$$

$$\tag{3.58}$$

We already know the eigenvalues, so we can use the Partial Fraction Expansion and Theorem of Residues where:

$$\Psi(s) = \frac{8 + 20i}{(32 + 128i)(s + 1 - 4i)} + \frac{-8 + 20i}{(-32 + 128i)(s + 1 + 4i)} + \frac{1}{2(s + 2)} - \frac{14}{17(s + 1)} \tag{3.59}$$

$$T(s) = \frac{14 - 4i}{(32 + 128i)(s + 1 - 4i)} + \frac{14 + 4i}{(32 - 128i)(s + 1 + 4i)} + \frac{1}{8(s + 2)} - \frac{2}{17(s + 1)} \tag{3.60}$$

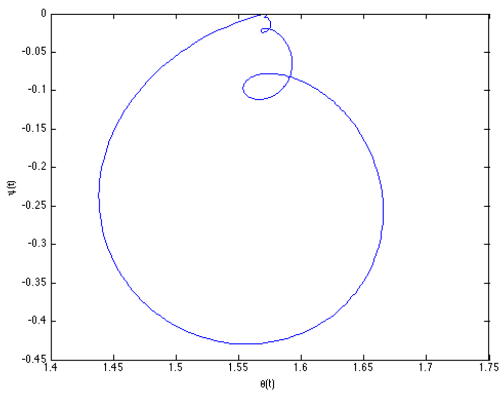
Taking the inverse Laplace Transform of $T(s)$ and $\Psi(s)$, we get

$$\psi(t) = \frac{8 + 20i}{32 + 128i}e^{(-1+4i)t} + \frac{-8 + 20i}{-32 + 128i}e^{(-1-4i)t} + \frac{1}{2}e^{-2t} - \frac{14}{17}e^{-t} \tag{3.61}$$

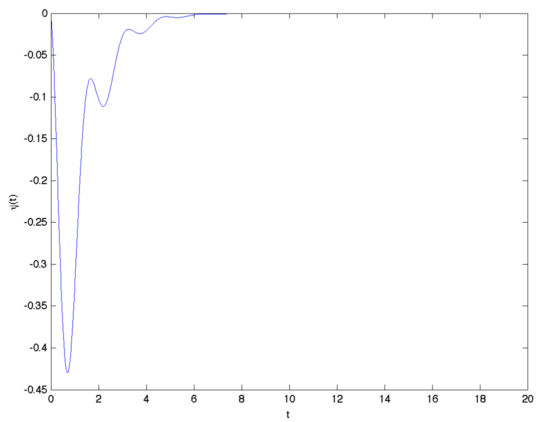
$$\tau(t) = \frac{14 - 4i}{32 + 128i}e^{(-1+4i)t} + \frac{14 + 4i}{32 - 128i}e^{(-1-4i)t} + \frac{1}{8}e^{-2t} - \frac{2}{17}e^{-t} \tag{3.62}$$

$$\theta(t) = \frac{\pi}{2} - \frac{14 - 4i}{32 + 128i}e^{(-1+4i)t} + \frac{14 + 4i}{32 - 128i}e^{(-1-4i)t} + \frac{1}{8}e^{-2t} - \frac{2}{17}e^{-t} \tag{3.63}$$

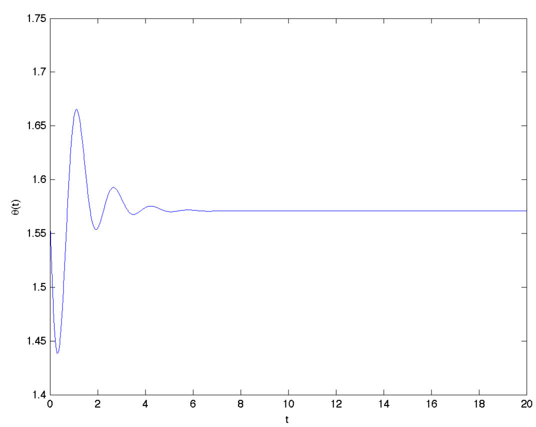
From figure 11(a), we can see that the solutions no longer oscillate as before. In figures 11(b) and 11(c), we can actually see that our choice of poles has caused the system to stabilize very rapidly, where $\lim_{t\to\infty}\psi(t) = 0$ and $\lim_{t\to\infty}\theta(t) \approx \frac{\pi}{2}$.



(a) θ vs. ψ



(b) t vs. ψ



(c) t vs. θ

Figure 11: Gyro-pendulum

4 Acknowledgements

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Appendices

5 Definitions

Definition 5.1. *The* ***state equations*** *of a system are a set of n first-order differential equations, where n is the number of independent states often written in matrix form as:*

$$\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t) \tag{5.1}$$

such that

- u - $m \times 1$ *input(control) vector*
- x - $n \times 1$ *state vector*
- $\boldsymbol{A}$ - $n \times n$ *system(plant) matrix*
- $\boldsymbol{B}$ - $n \times m$ *input(control) matrix*

. *The solution to the state equations are:*

$$x(t) = e^{\boldsymbol{A}t}x(0) + (e^{\boldsymbol{A}t}\boldsymbol{B}) \cdot u(t) \tag{5.2}$$

Definition 5.2. *A* ***state vector*** *is the set of state variables $x_i(t)$ that represent the elements or the components of the n-dimensional state vector $\vec{x}(t)$, i.e.*

$$\vec{x} \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \equiv \vec{x}$$

The system characteristic equation is of order n and the state equations consists of n first-order differential equations.

Definition 5.3. *The* ***state space*** *is defined as the n-dimensional space in which the components of the state vector represent its coordinate axes. If the space is two-dimensional, it is referred to as the phase plane.*

Definition 5.4. *The* ***Linear dynamic system*** *consists of the state equations and the* ***system outputs equations****, i.e.*

$$\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t), \qquad x(t_0) = x_0 \tag{5.3}$$
$$y(t) = \boldsymbol{C}x(t) + \boldsymbol{D}u(t) \tag{5.4}$$

such that

- ***C*** - $l \times n$ *output matrix*
- ***D*** - $l \times m$ *feedforward matrix*[1]
- y - $l \times 1$ *vector*

The system can be expressed as a multiple-input multiple-output (MIMO) system as above, or as a single-input single-output (SISO) system, where we will have

$$\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{b}u(t), \qquad x(t_0) = x_0 \tag{5.5}$$

$$y(t) = \boldsymbol{c}x(t) + \boldsymbol{d}u(t) \tag{5.6}$$

where $\boldsymbol{b}$,$\boldsymbol{c}$, *and* $\boldsymbol{d}$ *are vectors.*

Definition 5.5. *A vector $\vec{x}$ is an **equilibrium** of a dynamic system with an input $\vec{u}$ if it has the property that once the state reaches $\vec{x}$ it remains at $\vec{x}$ for all future time, t, i.e. $\dot{\vec{x}}(t) = 0$ as t approaches ∞.*

Updated May 13, 2009.

[1] For simplicity, the feedforward matrix,**D**, or the feedforward vector, **d**, will be equal to the zero matrix and zero vector, respectively, where feedforward feeds input directly to the output.